2-D Sedov Taylor Blast Wave

09 December 2019 12:43

But first: The Design Prize
Runner-Up: Sarina Wüthrich
Winner: Dominic Stalder

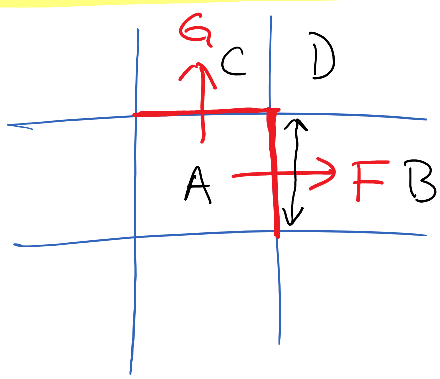
2D again:

$$\frac{\partial \underline{U}}{\partial t} + \nabla \cdot \underline{F}(\underline{U}) = 0 \qquad \underline{F} = \begin{bmatrix} F \\ G \end{bmatrix}$$

Piecewise constant states averaged over cell $\langle u \rangle_{i,j}^n$

$$\langle u \rangle_{i,j}^n = \frac{1}{\Delta x \Delta y} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} u(x, y, t^n)\, dx\, dy$$

$$\boxed{\langle u \rangle_{i,j}^{n+1} - \langle u \rangle^n + \frac{\Delta t}{\Delta x}\left[F_{i+\frac{1}{2},j}^{n+\frac{1}{2}} - F_{i-\frac{1}{2},j}^{n+\frac{1}{2}}\right] + \frac{\Delta t}{\Delta y}\left[G_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} - G_{i,j-\frac{1}{2}}^{n+1/2}\right] = 0}$$



$$F_{i+\frac{1}{2},j}^{n+1/2} = \frac{1}{\Delta t} \frac{1}{\Delta y} \int_{t^n}^{t^{n+1}} dt \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} dy\, F(x_{i+\frac{1}{2}}, y, t)$$

$$G_{i,j+\frac{1}{2}}^{n+\frac{1}{2}} = \frac{1}{\Delta t} \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} dt \int_{x_{i-\frac{1}{2}}}^{x_{i+1/2}} dx\, G(x, y_{j+\frac{1}{2}}, t)$$

2-D LAX Scheme:

$$U_{i,j}^{n+1} = \frac{1}{4}\left(\quad \right) - \frac{\Delta t}{2\Delta x}\left(F_{i+1,j}^n - F_{i-1,j}^n\right) - \frac{\Delta t}{2\Delta y}\left(G_{i,j+1}^n - G_{i,j-1}^n\right)$$

$$U = \begin{pmatrix} \rho \\ \rho u \end{pmatrix} \qquad \underline{F}(\underline{U}) = \begin{bmatrix} \rho u & \rho u^2 + P & \rho u v & u(E+P) \end{bmatrix}$$

$$\underline{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ E \end{pmatrix} \qquad \underline{F}(\underline{U}) = \begin{bmatrix} \rho u & \rho u^2 + P & \rho u v & u(E+P) \\ \rho v & \rho u v & \rho v^2 + P & v(E+P) \end{bmatrix}$$

$$\underline{E}(\underline{U}) = \begin{bmatrix} F \\ G \end{bmatrix}$$

F — Flux in x-direction

G — Flux in y-direction

$$E = \frac{1}{2}\rho(u^2+v^2) + \rho e$$

e — thermal energy per unit mass

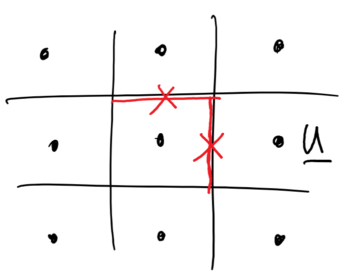
Introduce an equation of state (EOS)

$$e = \frac{P}{(\gamma-1)\rho} \qquad \text{Ideal Gas} \quad \gamma = \frac{f+2}{f} \simeq \frac{4}{2}$$

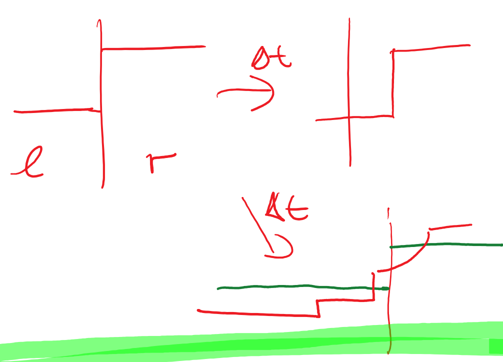
f degrees of freedom

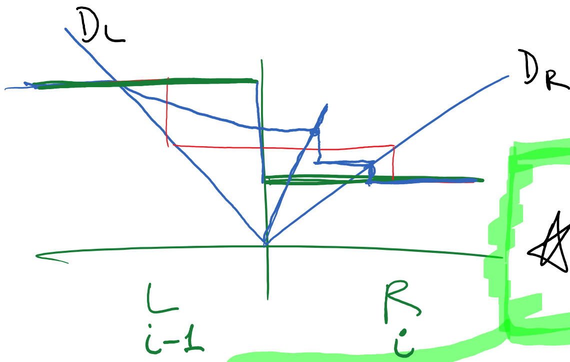
$$\boxed{\frac{\partial \underline{U}}{\partial t} + \nabla \cdot \underline{F}(\underline{U}) = 0}$$

How do we solve the fluxes (Approximate):



Riemann Problem:





☆ Lax-Friedrichs Riemann Solver

$$D_L = -D_R$$

$$F^n_{i-\frac{1}{2}} = \frac{1}{2}\left(F^n_{i-1} + F^n_i\right) - \frac{1}{2} D_{max}\left(U^n_i - U^n_{i-1}\right)$$

$\rightarrow D = |u| + C_s \leftarrow$ sound speed

$$D = |u| + C_s \leftarrow \text{sound speed}$$

Discontinuity Speed

$$C_s = \sqrt{\gamma \frac{P}{\rho}}$$

$$\boxed{D_{max} = \max(D_L, D_R)}$$

Corner Transport Upwind:

① Flux in x-directions

① $$F^{n+\frac{1}{2}}_{i-\frac{1}{2},j} = \frac{1}{2}\left({}^*F^{n+\frac{1}{2}}_{i-1,j} + {}^*F^{n+\frac{1}{2}}_{i,j}\right) - \frac{1}{2} D^u_{max}\left({}^*\underline{U}^{n+\frac{1}{2}}_{i,j} - {}^*\underline{U}^{n+\frac{1}{2}}_{i-1,j}\right)$$

also need $F^{n+1/2}_{i+\frac{1}{2},j}$!

$${}^*\underline{U}^{n+\frac{1}{2}}_{i,j} = \underline{U}^n_{i,j} + \frac{\Delta t}{2\Delta y}\left(G^n_{i,j+1/2} - G^n_{i,j-\frac{1}{2}}\right) \leftarrow \text{y-direction predictn step}$$

$$G^n_{i,j+1/2} = \frac{1}{2}\left(G^n_{i,j+1} + G^n_{i,j}\right) - \frac{1}{2} D^v_{max}\left(\underline{U}^n_{i,j+1} - \underline{U}^n_{i,j}\right)$$

$G^n_{i,j-\frac{1}{2}} =$ " " " "

---

② Flux in y-direction

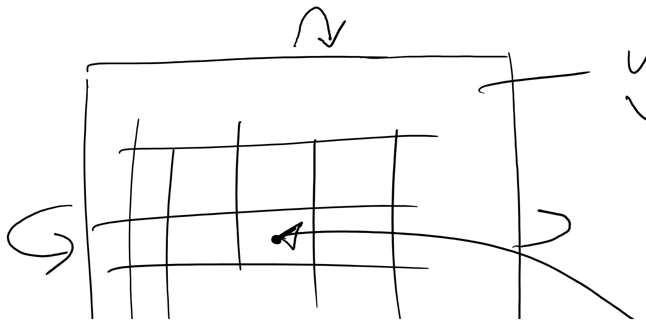
② $$G^{n+1/2}_{i,j-\frac{1}{2}} = \frac{1}{2}\left({}^*G^{n+1/2}_{i,j-1} + {}^*G^{n+\frac{1}{2}}_{i,j}\right) - \frac{1}{2} D^v_{max}\left({}^*U^{n+\frac{1}{2}}_{i,j} - {}^*U^{n+\frac{1}{2}}_{i,j-1}\right)$$

also need $G^{n+1/2}_{i,j+1/2}$!

$${}^*\underline{U}^{n+1/2}_{i,j} = \underline{U}^n_{i,j} + \frac{\Delta t}{2\Delta x}\left(F^n_{i+\frac{1}{2},j} - F^n_{i-\frac{1}{2},j}\right)$$

x-direction predictn step!

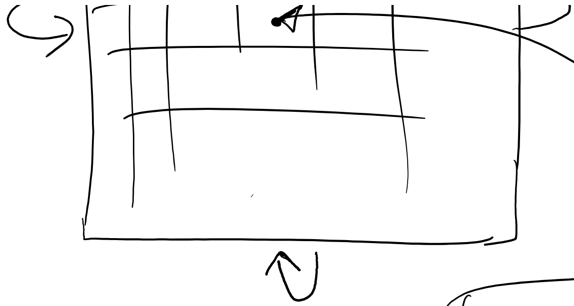
$$F^n_{i+\frac{1}{2},j} = \frac{1}{2}\left(F^n_{i+1,j} + F^n_{i,j}\right) - \frac{1}{2} D^u_{max}\left(\underline{U}^n_{i+1,j} - \underline{U}^n_{i}\right)$$



$u = 0 \quad \rho = 1$

$v = 0$

$P = 10^{-5} \rightarrow e = 10^{-5}$



$e = 1!$

Colella 1990