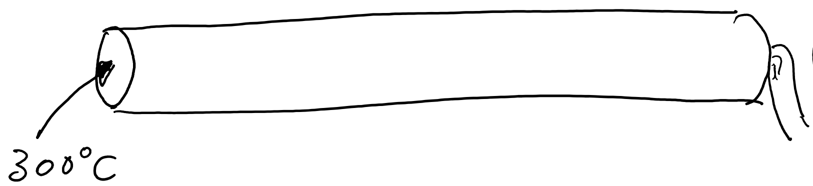
SINS 10: Diffusion Equation and Numerical Stability

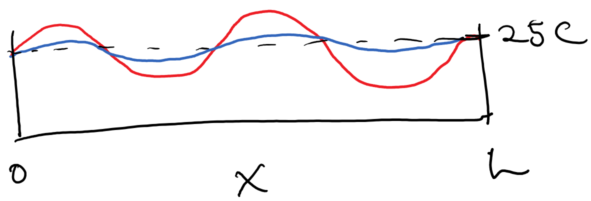
18 November 2019 13:01

$$\frac{\partial u}{\partial t} = D\nabla^2 u \qquad u(r,t) \qquad \text{Parabolic Case}$$

Diffusion Equation



How long can you hold your finger here



Should "smooth out" over time, not amplify.

$x \in [0,L]$ $t \geq 0$ Boundary and Initial Condition

$$u(t=0,x) = u^{(0)}(x)$$
$$u(t,x=0) = u_1(t)$$
$$u(t,x=L) = u_2(t)$$

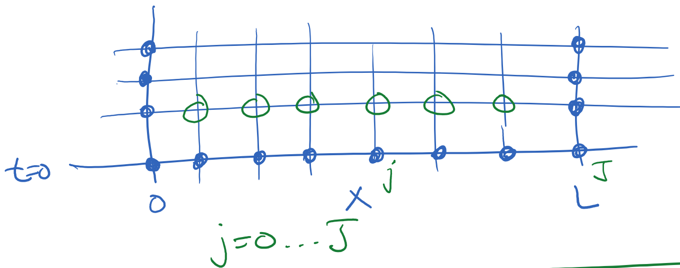
$$\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} \approx \frac{u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}}{\Delta x^2}$$

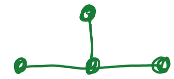
+1 -2 +1

Option 1: $\frac{\partial u}{\partial t} \simeq \frac{u_j^{(n+1)} - u_j^{(n)}}{\Delta t}$

$$u_j^{(n+1)} - u_j^{(n)} \approx \underbrace{\frac{D\Delta t}{\Delta x^2}}_{\alpha}\left(u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}\right)$$

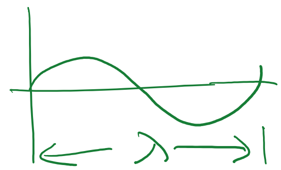
$$\boxed{u_j^{(n+1)} = \underbrace{u_j^{(n)} + \alpha\left(\bullet\!-\!\bullet\!-\!\bullet\right)^{(n)}}_{\text{everything is given}}}$$





von Neumann Stability Analysis

$$u_j^{(n)} = A^n e^{ikj\Delta x} \qquad Ae^{i\theta} = A(\cos\theta + i\sin\theta)$$



$$\lambda = \frac{2\pi}{k}$$

$|A| < 1$: Stable — the wave gets smoothed out $A \to 0$

$|A| > 1$: Unstable

$$u_j^{(n+1)} - u_j^{(n)} = \alpha\left(u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}\right)$$

$$A^{n+1} e^{ikj\Delta x} - A^n e^{ikj\Delta x} = \alpha A^n\left(e^{ik(j+1)\Delta x} - 2e^{ikj\Delta x} + e^{ik(j-1)\Delta x}\right)$$

pull out $\cancel{A^n e^{ikj\Delta x}}$: $(A-1) = \alpha\left(\underbrace{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}_{\frac{\cos(k\Delta x)-1}{2}}\right)$

$$\cos(\theta) \equiv \frac{e^{ik\Delta x} + e^{-ik\Delta x}}{2}$$

$$= 2\alpha\left(\cos(k\Delta x) - 1\right)$$

$$\boxed{A = 1 - 4\alpha \sin^2\left(\frac{k\Delta x}{2}\right)}$$

We want $|A| < 1$ for all possible k.

$$\sin^2(\,) \in [0, 1]$$

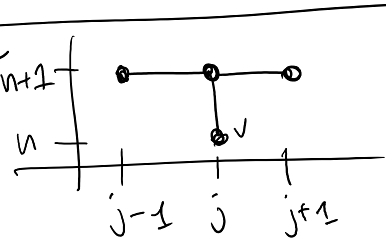
$$A \in 1 - 4\alpha[0,1] \in [1-4\alpha, \overset{\checkmark}{1}]$$

$$|A| < 1 \Rightarrow A^2 < 1 \quad \checkmark$$
$$\Rightarrow A \in [-1, 1]$$

lower bound is the critical one

$$-1 < 1 - 4\alpha$$
$$-2 < -4\alpha$$
$$\boxed{\frac{1}{2} > \alpha}$$
$$\boxed{\frac{D\Delta t}{\Delta x^2} < \frac{1}{2}\,!}$$
$$\boxed{\Delta t < \frac{(\Delta x)^2}{2D}}$$

Option 2



$$u_j^{(n+1)} = u_j^{(n)} + \alpha\left(u_{j+1}^{(n+1)} - 2u_j^{(n+1)} + u_{j-1}^{(n+1)}\right)$$

Implicit Method

$$A = \frac{1}{1 + 4\alpha \sin^2\left(\frac{k\Delta x}{2}\right)}$$

$$|A| \le 1 \quad \forall k \qquad \text{Always Stable!}$$

$$M\underline{x} = \underline{b} \qquad \begin{pmatrix} \ddots & \ddots & \emptyset \\ \ddots & \ddots & \ddots \\ \emptyset & \ddots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_{n-1} \end{pmatrix}$$

Error in these 2 Methods $\;\Theta(\Delta x^2)$
$\qquad\qquad\qquad\qquad\qquad\;\Theta(\Delta t)$

For high accuracy I need to take a lot of timesteps in both cases.

Option 3: Crank-Nicholson Method

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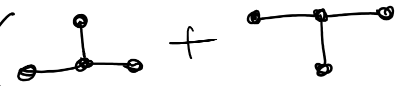
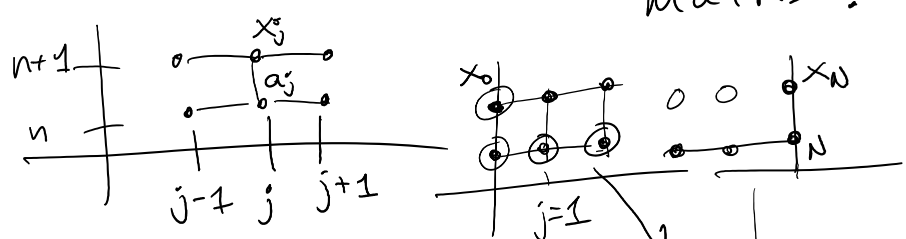
$\frac{1}{2}\Big($  $\Big)$ Implicit $\mathcal{O}(\Delta t^2)$

$$u_j^{(n+1)} - u_j^{(n)} = \frac{\alpha}{2}\left(u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)} + u_{j+1}^{(n+1)} - 2u_j^{(n+1)} + u_{j-1}^{(n+1)}\right)$$

$$\underline{\underline{M}}\,\underline{x} = \underline{b}$$

Solve this system where $\underline{\underline{M}}$ is a tridiagonal matrix.



$j=1$: $$(1+\alpha)x_1 - \frac{\alpha}{2}x_2 = \underbrace{\boxed{\frac{\alpha}{2}(x_0 + a_0 + a_2) + (1-\alpha)a_1}}_{b_1}$$

$j=2=N-2$: $$-\frac{\alpha}{2}x_{j-1} + (1+\alpha)x_j - \frac{\alpha}{2}x_{j+1} = \frac{\alpha}{2}a_{j-1} + (1-\alpha)a_j + \frac{\alpha}{2}a_{j+1}$$

$j=N-1$: $$-\frac{\alpha}{2}x_{N-2} + (1+\alpha)x_{N-1} = \frac{\alpha}{2}(x_N + a_{N-2} + a_N) + (1-\alpha)a_{N-1}$$

$$\begin{pmatrix} (1+\alpha) & -\frac{\alpha}{2} & \emptyset & \emptyset & \emptyset \\ -\frac{\alpha}{2} & (1+\alpha) & -\frac{\alpha}{2} & & \emptyset \\ & -\frac{\alpha}{2} & (1+\alpha) & -\frac{\alpha}{2} & \\ \emptyset & & \ddots & \ddots & -\frac{\alpha}{2} \\ & & & -\frac{\alpha}{2} & (1+\alpha) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \\ x_{N-1} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \\ b_{N-1} \end{pmatrix}$$

Tridiagonal Solution — 2N steps

Numerical Stability Experiment

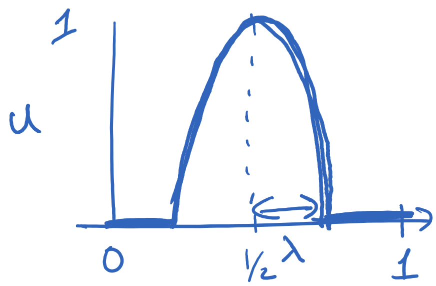
Numerical Stability Experiment

Let $D = 1$ $\quad L = x_{max} = 1$

Boundary $u = 0$ for $x = 0, 1$

with Initial Condition

$$u^{(0)}(x) = \begin{cases} 1 - \dfrac{(x-\frac{1}{2})^2}{\lambda^2}, & |x - \frac{1}{2}| < \lambda \\ 0, & \text{otherwise} \end{cases}$$



$N = 100$ Grid points $\quad \Delta x = \dfrac{1}{N}$

Δt: 4.9×10^{-5}, 5.0×10^{-5}, 5.1×10^{-5}

Use Option 1: