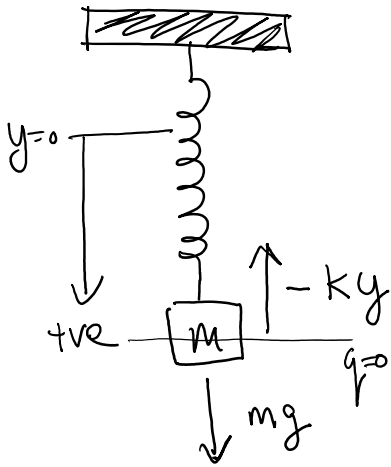


Harmonic Oscillator



$$F = mg - ky$$

to make it easier
 $m=1$ and $K=1$

$$F = g - y \quad \text{Define } q=0 \text{ where } F=0!$$

$$q = y - g = -F$$

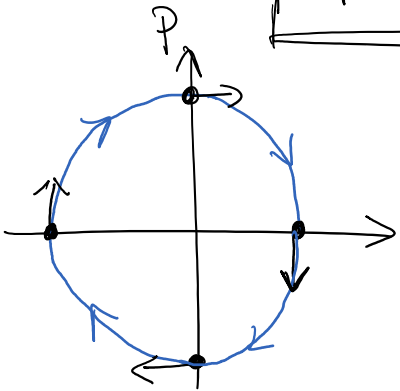
$$\dot{q} = \frac{dq}{dt}$$

Newton: $F = ma = m\ddot{y}$

$$\ddot{q} = \ddot{y} = F = -q$$

Momentum $p = mv = m\dot{y} = \dot{q}$

$$\begin{aligned} \dot{q} &= p \\ \dot{p} &= -q \end{aligned}$$



$$r^2 = p^2 + q^2$$

Should be conserved forever!

$$H = \underbrace{\frac{1}{2}p^2}_{\text{Kinetic Energy}} + \underbrace{\frac{1}{2}q^2}_{\text{Potential Energy}}$$

$$\dot{q} = \frac{\partial H}{\partial p}$$

Seperable
Hamiltonian

$$\dot{p} = -\frac{\partial H}{\partial q} \leftarrow \frac{1}{2}q^2$$

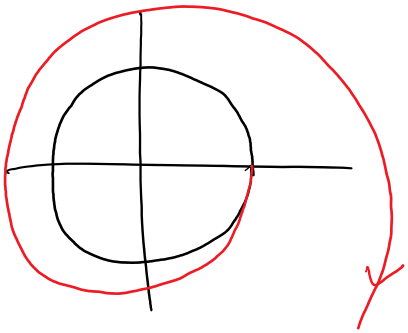
$$\dot{p} = - \frac{\partial H}{\partial q} = -\frac{\partial U}{\partial q}$$

What about for Forward-Euler Method

$$q_{n+1} = q_n + h p_n$$

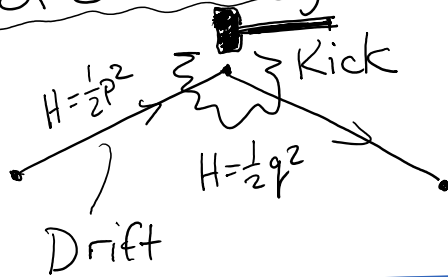
$$p_{n+1} = p_n - h \frac{\partial U}{\partial q}$$

$$q_{n+1}^2 + p_{n+1}^2 = \underbrace{(1+h^2)}_{1!} (q_n^2 + p_n^2)$$



BAD!

Symplectic Integrator



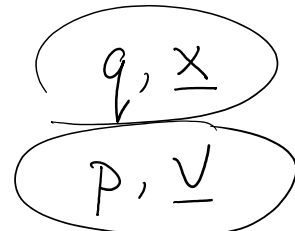
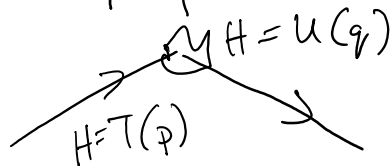
Leap-Frog
or Strömer-Verlet

$$H_0: \quad q_{n+\frac{1}{2}} = q_n + \frac{1}{2} h p_n \quad \text{"half Drift"}$$

$$p_{n+1} = p_n - h \frac{\partial U}{\partial q}(q_{n+\frac{1}{2}}) \quad \text{"Kick"}$$

$$q_{n+1} = q_{n+\frac{1}{2}} + \frac{1}{2} h p_{n+1} \quad \text{"Drift"}$$

$$H = T(p) + U(q)$$



$$H = T(p)$$

$$p, \underline{v}$$

$\mathcal{O}(h^3)$

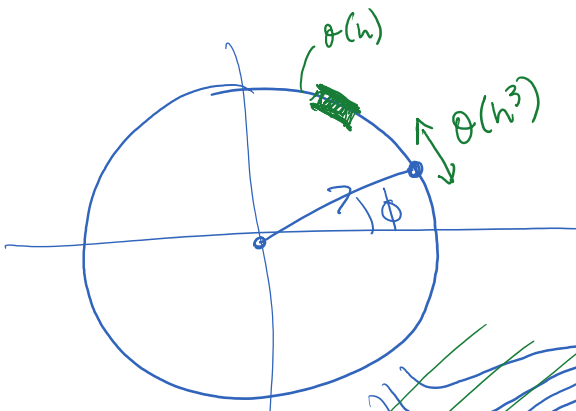
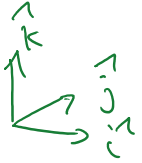
$$\underline{x}_{\frac{1}{2}} = \underline{x}_0 + \frac{1}{2} h \underline{v}_0$$

$$\underline{v}_1 = \underline{v}_0 + h \left(-\nabla u \left(\underline{x}_{\frac{1}{2}} \right) \right)$$

$$\underline{x}_1 = \underline{x}_{\frac{1}{2}} + \frac{1}{2} h \underline{v}_1$$

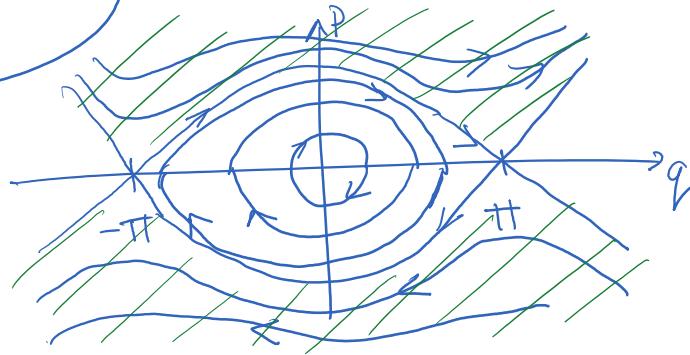
$$\equiv -\frac{\partial H}{\partial q}$$

$$\nabla u = \frac{\partial u}{\partial x} \hat{i} + \frac{\partial u}{\partial y} \hat{j} + \frac{\partial u}{\partial z} \hat{k}$$



Simple Pendulum

$$H = \frac{1}{2} p^2 - \epsilon \cos q \quad \text{angle}$$



Mid-pt.
Runge-Kutta
Symplectic
Leapfrog