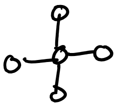
Recall: Jacobi Method on an $N \times N$ grid to solve the PDE:



$$\boxed{\nabla^2 \phi = 4\pi G \rho}$$

takes $\mathcal{O}(N^4)$ to converge to some accuracy in ϕ

SOR ideally takes $\mathcal{O}(N^3)$.

But now we have seen the FFT-Convolution technique takes $\mathcal{O}(N^2 \log(N))$ to machine precision.

$$-k^2 \phi_k = 4\pi G \rho_k$$

solve for ϕ_k

$\downarrow$ IFT

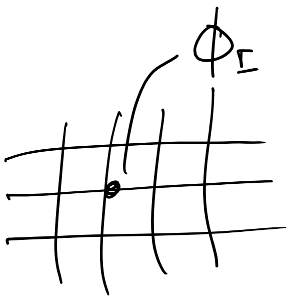
$\phi(\underline{r})$

Force : $\underline{F} = -\nabla \phi \cdot m$

$$= \underline{\underline{\hat{i} \frac{\partial}{\partial x}}} + \hat{j} \frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial x} \phi = ? = \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \phi_k e^{i \underline{k} \cdot \underline{r}} d\underline{k}$$

$$= \int_{-\infty}^{\infty} \phi_k \left[\frac{\partial}{\partial x} e^{i(k_x x + k_y y)} \right] dk_x dk_y$$



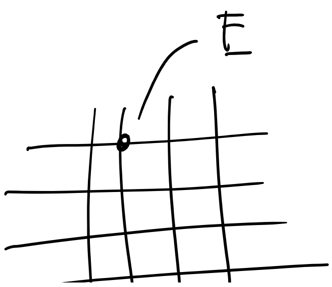
$$= i k_x \cdot \phi_k$$

recall $\phi_k = -\frac{4\pi G}{k^2} \rho_k$

then $F_{kx} = -i 4\pi G \frac{k_x}{k^2} \rho_k$

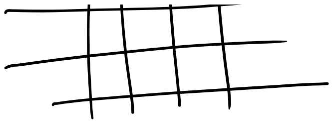
$\downarrow$ IFT

F_x



but also

$F_{ky} = -i 4\pi G \frac{k_y}{k^2} \rho_k$



$$F_{k_y} = -i\, 4\pi G \frac{i k_y}{k^2} \rho_k$$
$$\downarrow$$
$$F_y$$

Can we do even better?

$\mathcal{O}(N^2) \longrightarrow \mathcal{O}(N)$

$N \times N$ grid — Number of grid points

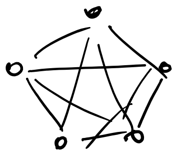
It is possible

Mass Assignment takes $\mathcal{O}(N)$ where N is the number of particles

Solving $\nabla^2 \phi = 4\pi G \rho$ by FFT $\rightarrow \mathcal{O}(M \log M)$ where M is the number of grid cells.

Magic 1

ϕ calculation



$\mathcal{O}(N^2)$ naively

multipole expansions get this down to

$\mathcal{O}(N \log N) \xrightarrow[\text{even}]{} \mathcal{O}(N)$

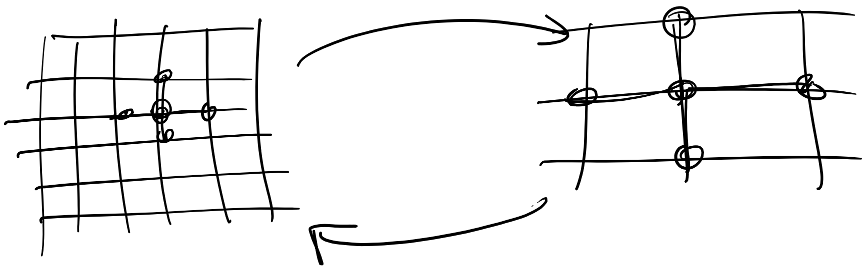
ϕ must be linear

Magic 2

$\mathcal{O}(N) \rightarrow$ mass assignment

$\mathcal{O}(M) \rightarrow$ solves the PDE

Multigrid Method



$$\nabla^2\phi = f(\phi) \quad ?$$