Diffusion Equation and Numerical Stability

16 November 2020 12:31

Parabolic PDE

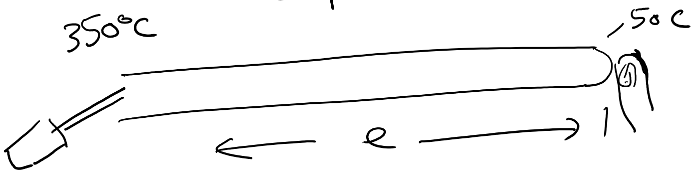
$$\frac{\partial T}{\partial t} = D \nabla^2 T$$

D — Diffusion Coefficient

Steel : $11 \ \frac{mm^2}{s}$

Silver : $\sim 100 \ \frac{mm^2}{s}$

Graphite : ~ 1000

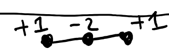


Should "smooth out" over time, not amplify

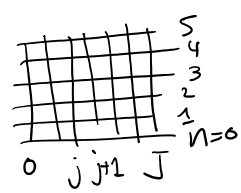
$x \in [0, L]$ $\quad t \geq 0$ $\quad$ Boundary and Initial Conditions.

$$u(t=0, x) = u^{(0)}(x)$$
$$u(t, x=0) = u_1(t)$$
$$\boxed{u(t, x=L) = u_2(t)}$$

$$\boxed{\nabla^2 u \equiv \frac{\partial^2 u}{\partial x^2} \approx \frac{u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}}{\Delta x^2}}$$



Option 1: $\quad \left.\frac{\partial u}{\partial t}\right| \cong \frac{u_j^{(n+1)} - u_j^{(n)}}{\Delta t}$

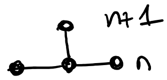


$$\frac{u_j^{(n+1)} - u_j^{(n)}}{\Delta t} = D \ \frac{u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}}{\Delta x^2}$$

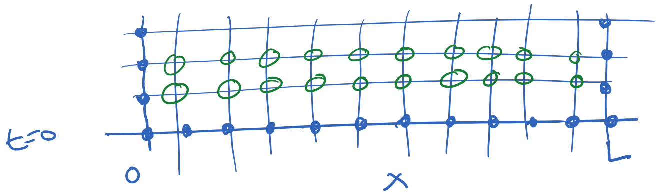
$$\boxed{\alpha = \frac{D \Delta t}{\Delta x^2}}$$

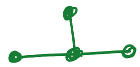
$$u_j^{(n+1)} = \underbrace{u_j^{(n)} + \alpha \left(\bullet\!-\!\bullet\!-\!\bullet \right)^{(n)}}_{\text{everything is given}}$$

n+1



everything is given
$\rightarrow$ explicit scheme



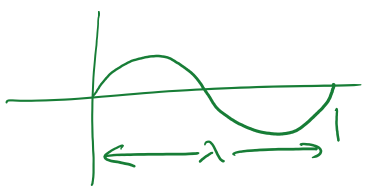


explicit

von Neumann Stability Analysis

$$\boxed{u_j^{(n)} = A^n e^{ikj\Delta x}}$$

$$Ae^{i\theta} = A(\cos\theta + i\sin\theta)$$



$$\lambda = \frac{2\pi}{k}$$

$|A| < 1$: Stable — the wave gets smoothed out $A^n \to 0$

$|A| > 1$: Unstable — Amplification!

$$u_j^{(n+1)} = u_j^{(n)} + \alpha\left(u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}\right)$$

$$A^{n+1}e^{ikj\Delta x} = \cancel{A^n e^{ikj\Delta x}} + \alpha A^n\left(e^{ik(j+1)\Delta x} - 2e^{ikj\Delta x} + e^{ik(j-1)\Delta x}\right)$$

$$A = 1 + \alpha\left(\underline{e^{ik\Delta x}} - 2 + \underline{e^{-ik\Delta x}}\right)$$

$$\cos(k\Delta x) = \frac{e^{ik\Delta x} + e^{-ik\Delta x}}{2}$$

$$= 1 + \alpha\left(2\cos(k\Delta x) - 2\right)$$

$$= 1 + 2\alpha\left(\cos(k\Delta x) - 1\right)$$

$$\sin^2\left(\frac{x}{2}\right) = \frac{1}{2}\left[1 - \cos(x)\right]$$

$$\boxed{A = 1 - 4\alpha\sin^2\left(\frac{k\Delta x}{2}\right)}$$

$$\boxed{A = 1 - 4\alpha \sin^2\left(\frac{k\Delta x}{2}\right)}$$

We want $|A| < 1$ for all possible k!

$$\sin^2(\,) \in [0, 1]$$

$$A \in 1 - 4\alpha[0, 1]$$

$$\in [1 - 4\alpha, 1]$$

$$|A| < 1 \Rightarrow A^2 < 1$$

$$\Rightarrow A \in [-1, 1]$$

lower bound is the critical one...

$$-1 < 1 - 4\alpha$$

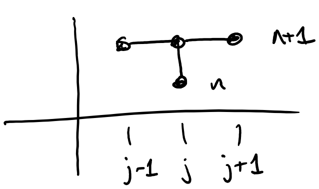
$$-2 < -4\alpha$$

$$\boxed{\alpha < \frac{1}{2}} \qquad \boxed{\frac{D\Delta t}{\Delta x^2} < \frac{1}{2}}$$

$$\boxed{\Delta t < \frac{(\Delta x)^2}{2D}}$$

Can we make a method that is unconditionally stable? Yes.

Option 2:



$$\underline{\underline{u_j^{(n+1)}}} = u_j^{(n)} + \alpha\left(\underline{\underline{u_{j+1}^{(n+1)}}} - 2\underline{\underline{u_j^{(n+1)}}} + \underline{\underline{u_{j-1}^{(n+1)}}}\right)$$

Implicit Method

$$A = \frac{1}{1 + 4\alpha \sin^2\left(\frac{k\Delta x}{2}\right)}$$

$|A| < 1 \;\forall k$ Always Stable!

$$M x = b \qquad \left(\diagdown\diagdown\diagdown \;\; 0\right) \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}^{(n+1)} \qquad \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

$$\underline{M}\,\underline{x} = \underline{b} \qquad \begin{pmatrix} \ddots & & \emptyset \\ & \ddots & \\ \emptyset & & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \end{pmatrix}$$

Error in these methods

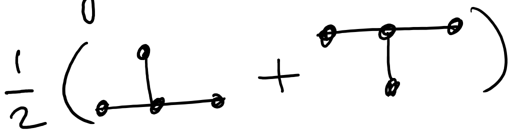
$\Theta(\Delta x^2)$ resolution

$\Theta(\Delta t)$ time resolution

For high accuracy I need to take a lot of timesteps in both cases.

Option 3: Crank-Nicholson Method

"Average" option 1 & 2 :



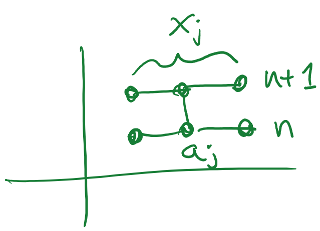
Implicit, Stable

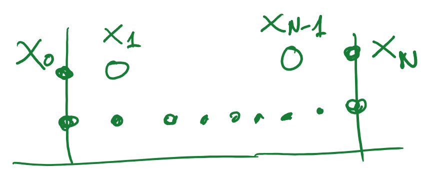
$\Theta(\Delta t^2)$!

$$\boxed{u_j^{(n+1)} - u_j^{(n)} = \frac{\alpha}{2}\left(u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)} + u_{j+1}^{(n+1)} - 2u_j^{(n+1)} + u_{j-1}^{(n+1)}\right)}$$

$$\underline{\underline{M}}\,\underline{x} = \underline{b}$$

Solve this system where $\underline{\underline{M}}$ is a tridiagonal Matrix.





$j = 1$: $(1-\alpha)x_1 - \frac{\alpha}{2}x_2 = \underbrace{\boxed{\frac{\alpha}{2}(x_0 + a_0 + a_2) + (1-\alpha)a_1}}_{b_1}$

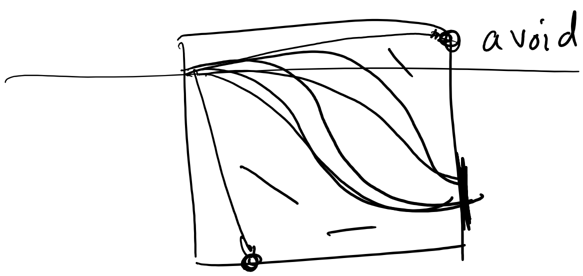
$j = 2..N-2$: $-\frac{\alpha}{2}x_{j-1} + (1-\alpha)x_j - \frac{\alpha}{2}x_{j+1} = \frac{\alpha}{2}a_{j-1} + (1-\alpha)a_j + \frac{\alpha}{2}a_{j+1}$

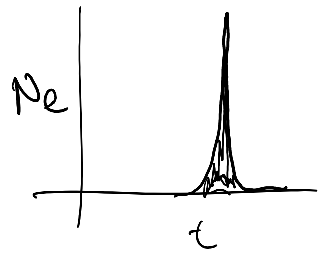
$j = N-1$: $-\frac{\alpha}{2}x_{N-2} + (1-\alpha)x_{N-1} = \underbrace{\boxed{\frac{\alpha}{2}(x_N + a_{N-2} + a_N) + (1-\alpha)a_{N-1}}}_{b_{N-1}}$

$$\begin{pmatrix} (1-\alpha) & -\frac{\alpha}{2} & & \emptyset \\ -\frac{\alpha}{2} & (1-\alpha) & -\frac{\alpha}{2} & \\ & & \ddots & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ \vdots \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\alpha}{2} & (1-\alpha) & -\frac{\alpha}{2} & & \emptyset \\ & -\frac{\alpha}{2} & (1-\alpha) & \ddots & \\ & \emptyset & & \ddots & -\frac{\alpha}{2} \\ & & & -\frac{\alpha}{2} & (1-\alpha) \end{pmatrix} \begin{pmatrix} x_2 \\ \vdots \\ x_{N-1} \end{pmatrix} = \begin{pmatrix} b_2 \\ \vdots \\ b_{N-1} \end{pmatrix}$$

Tridiagonal Solution   2N steps





PRIZE! $\boxed{-1\text{kV to } 1\text{kV}}$