Smoothed Particle Hydrodynamics (SPH)

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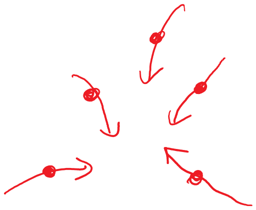
AMR

Adaptive Mesh Refinement

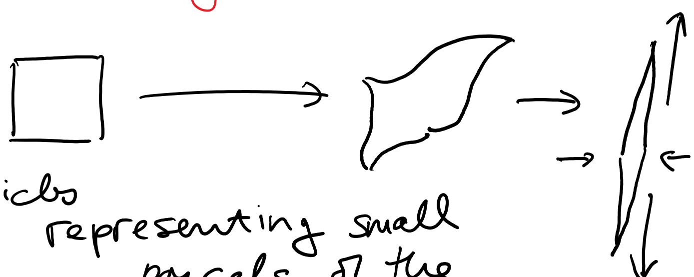
↳ higher resolution in the Eulerian Methods



Lagrangian Methods


higher density region

2D or higher is not easy



SPH:

Particles representing small parcels of the fluid

$$\frac{d\underline{u}}{dt} \quad \text{"Time derivative follows the motion"} \equiv \frac{\underline{u}(\underline{x}+\underline{u}\delta t, t+\delta t) - \underline{u}(\underline{x},t)}{\delta t}$$

use a Taylor expansion

$$\frac{dF}{dt} = \frac{1}{\delta t}\left\{ \underbrace{F(\underline{x}+\underline{u}\delta t, t+\delta t)} - F(\underline{x},t) \right\}$$

$$= \frac{1}{\delta t}\left\{ \cancel{F} + \frac{\partial F}{\partial x} u\delta t + \frac{\partial F}{\partial y} v\delta t + \frac{\partial F}{\partial z} w\delta t + \frac{\partial F}{\partial t}\delta t - \cancel{F} \right\}$$

$\underline{u} \equiv \langle u, v, w \rangle$

$\underline{x} \equiv \langle x, y, z \rangle$

$$\boxed{\frac{dF}{dt} = (u \cdot \nabla) F + \frac{\partial F}{\partial t}}$$

Convective derivative

$$\boxed{\frac{dF}{dt} = (\underline{u}\cdot\nabla)F + \frac{\partial F}{\partial t}}$$ Convective derivative

$$\frac{d\underline{u}}{dt} = \underline{u}\cdot\nabla\underline{u} + \frac{\partial \underline{u}}{\partial t}$$

$m \cdot \underline{a} = \underline{F}$ ← external force

small volume $\Delta V$

$$\rho \cdot \Delta V \frac{d\underline{u}}{dt} = \underline{F} \sim mg$$

$$\rho \cdot \cancel{\Delta V}\, \underline{u}\cdot\nabla\underline{u} + \rho \cancel{\Delta V} \frac{\partial u}{\partial t} = \rho \cancel{\Delta V} g$$

$$\rho \frac{\partial u}{\partial t} + \rho\, \underline{u}\cdot\nabla\underline{u} = \rho g$$

does this describe the transfer of momentum?

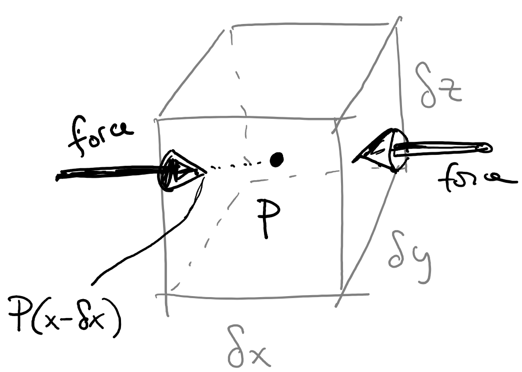
Is this a valid equation of motion?

No not quite

We are missing the pressure contribution to momentum transfer!

$$\frac{d\underline{u}}{dt} = g - \frac{\nabla P}{\rho}$$

Euler equation: $\boxed{\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = g - \frac{\nabla P}{\rho}}$




$P$, isotropic

in +ve x-direction: $\left\{P - \frac{1}{2}\frac{\partial P}{\partial x}\cdot\delta x\right\}\delta y\,\delta z$

Pressure is force per unit Area

in −ve x-direction: $\left\{P + \frac{1}{2}\frac{\partial P}{\partial x}\delta x\right\}\delta y\,\delta z$

$$= -\frac{\partial P}{\partial x}\delta x\,\delta y\,\delta z$$

this is the net force on the cube due to the pressure ... in the x-direction.

cube due to the pressure gradient in the x-direction.

For just x we get:

$$\rho \cancel{\delta x \delta y \delta z} \cdot \frac{du}{dt} = \rho \cancel{\delta x \delta y \delta z}\, g_x - \frac{\partial P}{\partial x} \cancel{\delta x \delta y \delta z}$$

$$\frac{du}{dt} = g_x - \frac{1}{\rho}\frac{\partial P}{\partial x}$$

do the same for the y and z directions

Generalize

$$\frac{du}{dt} = g - \frac{\nabla P}{\rho}$$

$$\boxed{\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = g - \frac{\nabla P}{\rho}}$$

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SPH - use particles to follow the flow and consider the various integrals in the conservation laws and use interpolations for these.

Consider an integral interpolant

$$A_I(\underline{r}) = \int_{\text{All Space}} A(\underline{r}')\, W(\underline{r}-\underline{r}'; h)\, d\underline{r}'$$

Kernel

1st $\int W \, d\underline{r} = 1$

2nd $\lim_{h \to 0} W(\underline{r}-\underline{r}'; h) = \delta(\underline{r}-\underline{r}')$

approximate this via summation

$$A_S(r) = \sum_b m_b \frac{A_b}{\rho_b} W(\underline{r}-\underline{r}_b, h)$$

Also appoximate its gradient

$$\nabla A_s(r) = \sum_b m_b \frac{A_b}{\rho_b} \nabla W(\underline{r}-\underline{r}_b, h)$$

although higher accuracy is obtained

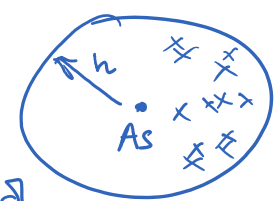
$$\rho \nabla A = \nabla(\rho A) - A \nabla \rho$$

Kernels: $$W(x,h) = \frac{1}{h\sqrt{\pi}} e^{-\left(\frac{x^2}{h^2}\right)}$$

fine but we want "compact" Kernel that goes over some local region.

What is the error in the approximation of $A_I$ by $A_s$? Depends a lot on how disordered the particles are good (typically) $\mathcal{O}(h^2)$

BAD:




Ignore points outside of the Kernel in the computation of $A_s$!

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What variables do we already have

$\underline{u}$, $P$, and $\rho$ 5-variables

so we need at least 2 more equations to close the system.

Special Cases:

isothermal flow: $P \propto \rho$ $\longrightarrow$ only 1 more equation

incompressible flow: $\rho$ is given

What are these equations?

[illegible] of mass

What are these equations.

1. Continuity Equation : conservation of mass
2. Energy Equation: exchange of energy between a fluid parcel and its environment.

Consider again the infinitesimal cube
net inflow in +ve x-direction

$$\left\{\rho u - \frac{1}{2}\frac{\partial(\rho u)}{\partial x}\delta x\right\}\delta y\,\delta z$$

net inflow $\Rightarrow -\dfrac{\partial(\rho u)}{\partial x}\delta x\,\delta y\,\delta z$

Total net inflow $= -\nabla\cdot(\rho\underline{u})\,\delta V$

$$\boxed{\frac{\partial\rho}{\partial t} + \underbrace{\nabla\cdot(\rho\underline{u})}_{\underline{u}\cdot\nabla\rho + \rho\nabla\cdot\underline{u}} = 0}$$

Continuity Equation

$$\boxed{\frac{d\rho}{dt} + \rho\nabla\cdot\underline{u} = 0}$$

Lagrangian Formulation of the Continuity equation.

For incompressible fluid this term is zero → Derivative of $\rho$ following the flow is also zero.