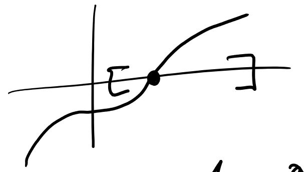
# Newton's Method and solving Kepler's Equation

Monday, 26 September 2022 11:09

$$f(x) = 0$$



$$|a-b| < \varepsilon_{\text{absolute}}$$

$$\frac{|a-b|}{|c|} < \varepsilon \text{ relative}$$

bits in mantissa

$$x = 1.01\underset{0}{1}\,\underset{1}{0}\,\underset{2}{1}\,\underset{3}{0}\,\underset{4}{1}$$

iteration count

Float: 23 bits mantissa $\Longrightarrow$ 23 iterations of bisection

constant number of bits (in this case 1) per iteration $\longrightarrow$ linearly convergent

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Yes. Quadratic Convergence double number of bits with each iteration.

Newton's Method

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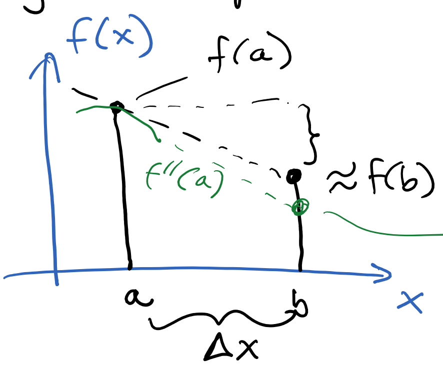
Taylor expansion




$$f'(a) \equiv \left.\frac{\partial f(x)}{\partial x}\right|_a$$

$$f(b) \approx f(a) + \Delta x \cdot f'(a)$$

$$f(a) + \Delta x \cdot f'(a) + \frac{\Delta x^2}{2} \cdot f''(a)$$

$$f(b) \equiv f(a+\Delta x) \cong f(a) + \Delta x\, f'(a) + \tfrac{1}{2}\Delta x^2 f''(a)$$

$$+ \tfrac{1}{6}\Delta x^3 f^{(3)}(a) + \cdots + \tfrac{1}{n!}\Delta x^n f^{(n)}(a)$$

$$+ \left[\frac{\Delta x^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(a + t\Delta x)\,dt\right]$$

Sometimes it is possible to find a practical and relatively tight upper bound to this integral.

$$f(x) = f(a+\Delta x) \cong f(a) + \Delta x \cdot f'(a)$$

Let's suppose this has a root at $x$

$$f(x) = 0$$

$$x = a + \Delta x$$

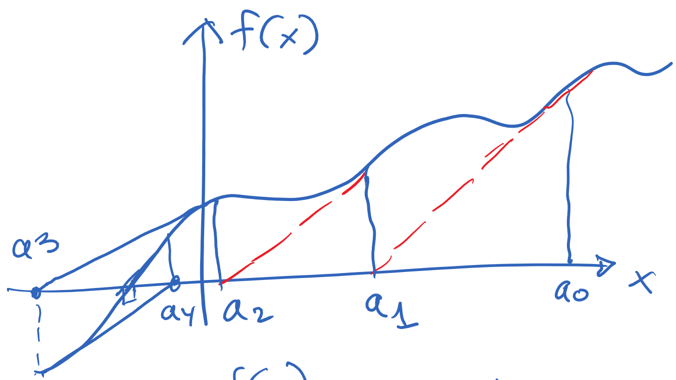
$$f(a) + \Delta x\, f'(a) \cong 0$$

$$\Delta x \cong -\frac{f(a)}{f'(a)}$$

$$\boxed{x \cong a - \frac{f(a)}{f'(a)}}$$

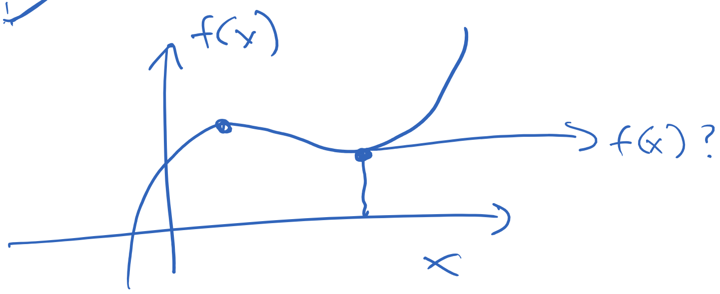
need this

$$\boxed{a = x}$$

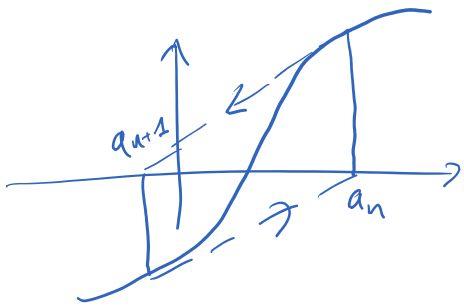



first guess $a_0$

Quadratic convergence when it is working




"escapes to $\infty$"




"Cycle"

sometimes it can be assumed that Newton's Method will converge.

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\# Operations: $\otimes$, $\oplus$, $\sqrt{x}$, $a/b$ division, $\sin x$

$$\sqrt{x} \rightarrow \left(\frac{1}{\sqrt{x}}\right) \cdot x \qquad \frac{a}{b} \rightarrow a \cdot \frac{1}{\sqrt{b}} \cdot \frac{1}{\sqrt{b}}$$

$$y = \frac{1}{\sqrt{x}}$$ y - is the unknown
x - is given!

$$0 = f(y) = y - \frac{1}{\sqrt{x}}$$

This is not so great $y^2 = \frac{1}{x}$

$$f(y) = x - \frac{1}{y^2}$$ weird, but it works...

$$y \leftarrow y - \frac{x - \frac{1}{y^2}}{2\frac{1}{y^3}}$$

$$y \leftarrow y - \frac{1}{2} x y^3 + \frac{1}{2} y$$

$$y \leftarrow y*(1.5 - 0.5*x*y*y)$$

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$$x = +M \times 2^e \qquad 1 \leq M < 2$$

$$\frac{1}{\sqrt{x}} = \underbrace{\frac{1}{\sqrt{M}}}_{} \times 2^{-e/2} \checkmark$$

$$\frac{\sqrt{2}}{2} < y < 1$$

$\approx 0.8535$

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Today's Assignment: Solve Kepler's Equation and plot it.

mean anomaly

$$M = E - e \sin E$$

eccentric anomaly

angle

a different angle

eccentricity

$$E(M, e) = ?$$

$f(E) = 0$ !

$$0 = E - e\sin E - M$$

$f(E)$

$f'(E)$

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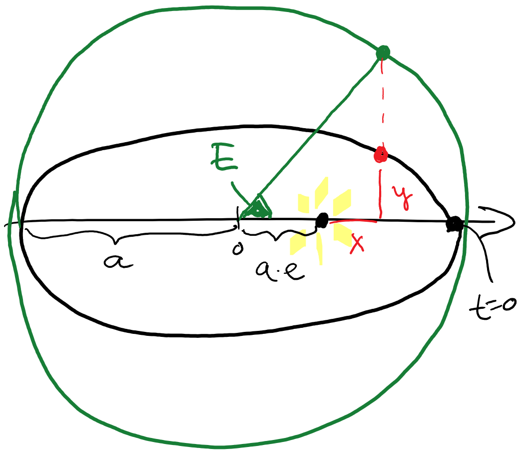
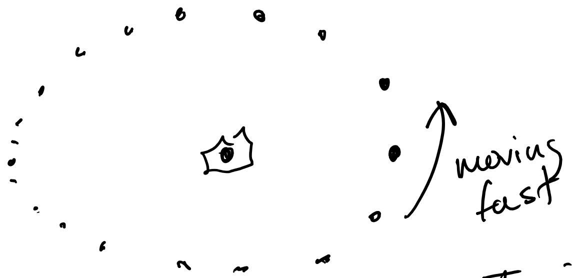
$M = n \cdot t \qquad n = \frac{2\pi}{T}$ — $T$ period in years

$T = a^{3/2}$ — $a$ AU




$$\boxed{\begin{aligned} x &= a \cdot \cos(E) - ae \\ y &= b \cdot \sin(E) \\ y &= a\sqrt{1-e^2}\,\sin(E) \end{aligned}}$$




$e = 0.5 \qquad a = 1$

$f'(E) = ?$

Initial $E_0 = M$