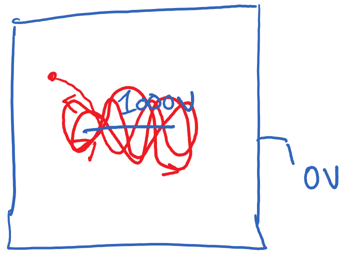
Bi-linear and Bi-cubic Interpolation: Electron Beams

15 November 2021 11:57

$\phi \quad \nabla\phi = a$

$\underline{F} = m\,\underline{a}$

PDE (→ $\underline{F}$) ODE (→ $\underline{a}$)



Electrodynamics

$$\underline{F} = e(\underline{E} + v_e \times \underline{B})$$

ignore this (→ $v_e \times \underline{B}$)

$\underline{E} \leftarrow -\nabla\Phi$

$$\underline{a} = \left(\frac{e}{m_e}\right)\left(-\nabla\Phi\right)$$

ODE

$$\ddot{x} = \dot{v}_x = \left(\frac{e}{m_e}\right)\left(-\frac{\partial\Phi}{\partial x}\right)$$

$$\ddot{y} = \dot{v}_y = \left(\frac{e}{m_e}\right)\left(-\frac{\partial\Phi}{\partial y}\right)$$

$$\dot{x} = v_x$$

$$\dot{y} = v_y$$

4 first order equations

$e = 1.60 \times 10^{-19}\,C$

$m_e = 9.11 \times 10^{-31}\,kg$

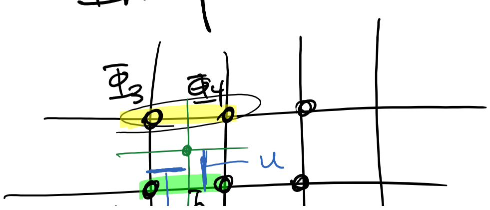
$$\left(\frac{e}{m_e}\right) = 1.76 \times 10^{11}\,C/kg$$

$$\Phi = \left[\frac{N\,m}{C}\right] \qquad \nabla\Phi = \left[\frac{N\,\cancel{m}}{C\,\cancel{m}}\right]$$

$$\left(\frac{e}{m_e}\right)\left(-\nabla\Phi\right) = \left[\frac{C}{kg}\right]\left[\frac{N}{C}\right] = \left[\frac{N}{kg}\right] = \left[\frac{\cancel{kg}\,m\,s^{-2}}{\cancel{kg}}\right]$$

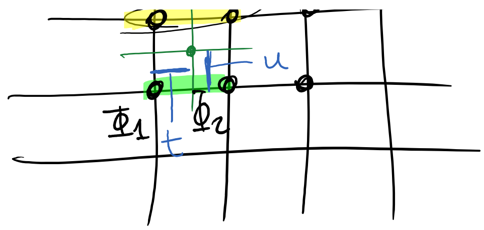
$$\ddot{x} = \left[m\,s^{-2}\right] \checkmark$$

Interpolation

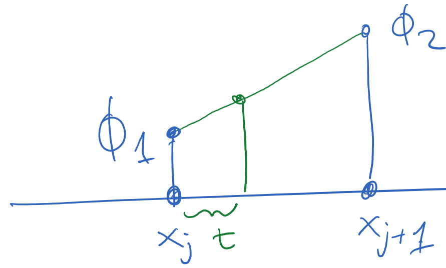


$$t = \frac{(x - x_j)}{(x_{j+1} - x_j)} = \frac{1}{\Delta}(x - x_j)$$

$$u = \frac{1}{\Delta}(y - y_e)$$



$$u = \frac{1}{\Delta}(y - y_e)$$



if $t=0$ $\phi = \phi_1$

if $t=1$ $\phi = \phi_2$

$$\phi = t\,\phi_2 + (1-t)\,\phi_1$$

$u=0$: $\phi(x, y_e) = (1-t)\phi_1 + t\phi_2$

$u=1$: $\phi(x, y_{e+1}) = (1-t)\phi_3 + t\phi_4$

$$\Phi(x,y) = (1-t)(1-u)\phi_1 + t(1-u)\phi_2 + (1-t)u\phi_3 + tu\phi_4$$

2-D linear interpolation

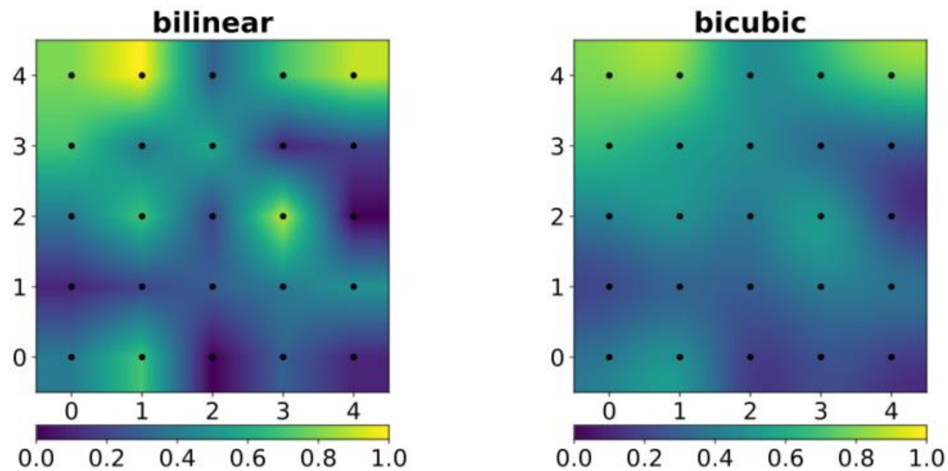
$$\left.\frac{\partial \Phi}{\partial x}\right|_u = \frac{\partial t}{\partial x}\cdot\frac{\partial \Phi}{\partial t} = \frac{1}{\Delta}\left[-(1-u)\phi_1 + (1-u)\phi_2 - u\phi_3 + u\phi_4\right]$$

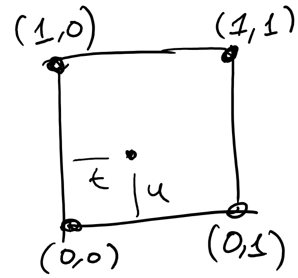
$$= \frac{1}{\Delta}\left[(1-u)(\phi_2 - \phi_1) + u(\phi_4 - \phi_3)\right]$$

leave as exercise: $\left.\frac{\partial \Phi}{\partial y}\right|_t = \ldots ?$

$$\dot{x} = v_x$$
$$\dot{v}_x = \left(\frac{e}{m_e}\right)\left[-\frac{1}{\Delta}\left((1-u)(\phi_2 - \phi_1) + u(\Phi_4 - \Phi_3)\right)\right]$$
$$\dot{y} = v_y$$
$$\dot{v}_y = \left(\frac{e}{m_e}\right)\left[-\frac{1}{\Delta}(\cdots)\right]$$

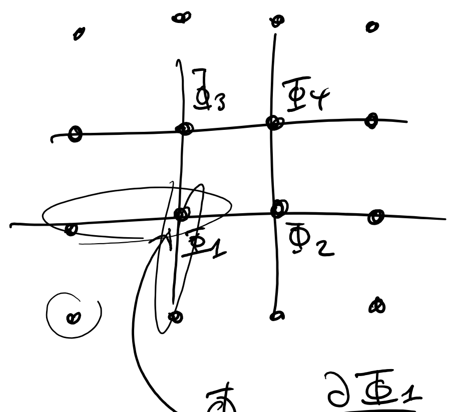
RK4
Midpoint
Leapfrog





$$\Phi(t,u)=\sum_{i=0}^{3}\sum_{j=0}^{3} a_{ij}\, t^i u^j$$

↑ 16 coefficients



Now we have 16 numbers to set the 16 coefficients of a_{ij}

$$\Phi_1,\ \frac{\partial \Phi_1}{\partial x},\ \frac{\partial \Phi_1}{\partial y},\ \frac{\partial^2 \Phi_1}{\partial x \partial y}$$

$\Phi(0,0)$ $\quad \Phi_x(0,0)$ $\quad \Phi_y(0,0)$ $\quad \Phi_{xy}(0,0)$

$$M=\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 0 & 3 & -2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & \ddots & \ddots \\ a_{20} & \ddots & & \\ a_{30} & & & \end{bmatrix} = M^T \cdot \begin{bmatrix} \Phi(0,0) & \Phi(0,1) & \Phi_y(0,0) & \Phi_y(0,1) \\ \Phi(1,0) & \Phi(1,1) & \Phi_y(1,0) & \Phi_y(1,1) \\ \Phi_x(0,0) & \Phi_x(0,1) & \Phi_{xy}(0,0) & \Phi_{xy}(0,1) \\ \Phi_x(1,0) & \Phi_x(1,1) & \Phi_{xy}(1,0) & \Phi_{xy}(1,1) \end{bmatrix} \cdot M$$

$$\Phi(x,y) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \cdot \begin{bmatrix} a_{ij} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ u \\ u^2 \\ u^3 \end{bmatrix}$$

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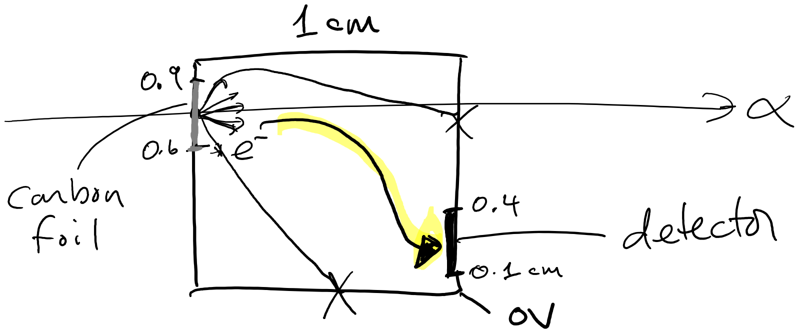
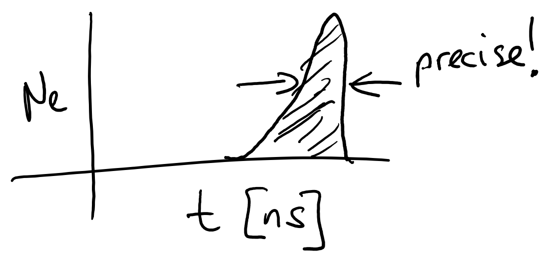
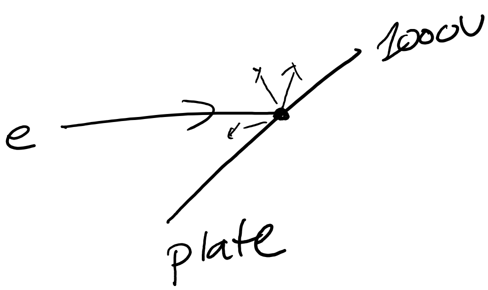
2 Weeks! ⟶ Real Prize for the winner!

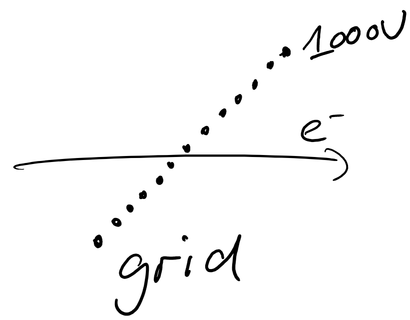
Design Prize



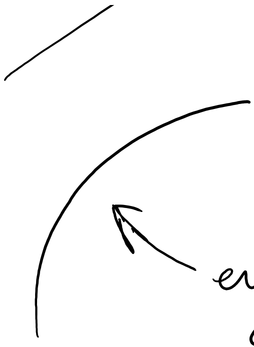


$|v_e| = 10^6$ m/s at random angles



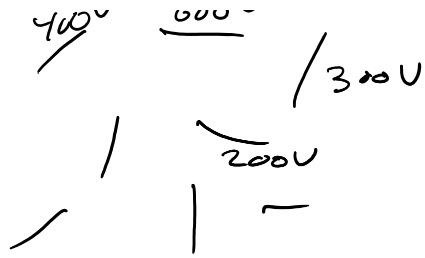


for us they are the same! Only electons can pass through the grid and rot the plate.



harder

engineering considerations



messy design
(not elegant)
too plates/grids