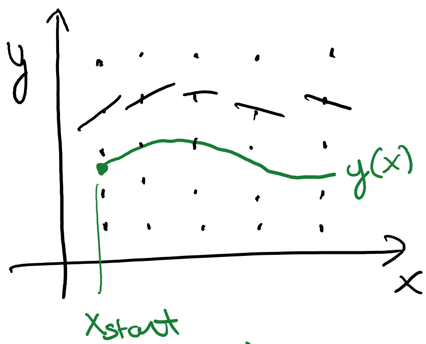
Ordinary Differential Equations

18 October 2021 12:54

$$\frac{d\,y(x)}{dx} = \underline{f}(x, y)$$



$$\frac{dy}{dx} = f(x,y)$$

$$y(x) = ?$$

$y(x_{start}) = y_{start}$ ← given

Need initial conditions (if $\frac{d}{dt}$)

boundary — $\frac{d}{dx}$

$y_i(x_{start}) = y_i^{start}$ Dirchlet Boundary Condition

$\frac{dy_i}{dx}(x_{start})$ = Slope is given somewhere

von Neumann Boundary Condition

N-equations ⟶ needs N boundary Conditions.

Analytical solutions can be difficult.

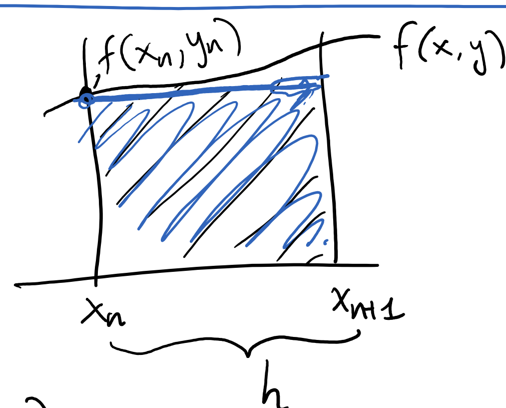
Numerical solution (algorithm) is "easy".

↳ Numerical errors can magnify, must be kept under control/tracked.

$$\int_{y_n}^{y_{n+1}} dy = \int_{x_n}^{x_{n+1}} f(x, y)\, dx$$

$$y_{n+1} - y_n = \int_{x_n}^{x_{n+1}} f(x_n, y_n)\, dx$$

$$= (x_{n+1} - x_n) \cdot f(x_n, y_n)$$

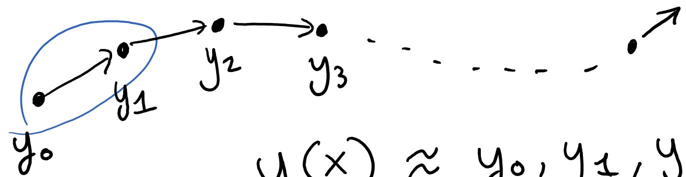


n

$$= (x_{n+1} - x_n) \cdot f(x_n, y_n)$$

$$\boxed{y_{n+1} = y_n + h \cdot f(x_n, y_n)}$$

Forward Euler Method



$$y(x) \approx y_0, y_1, y_2, \cdots$$

Error depends on the stepsize h

Truncation Error: Error associated with the algorithm or method, and not the precision of the floating point calculations.

Local error: Error over a single step

Global error: Error over a fixed interval

Error of Forward Euler:

$$\int_{x_n}^{x_{n+1}} f(x,y)\,dx \qquad f(x, y(x)) = f(x, \underbrace{y(x_n + h)}_{\text{Taylor expand}})$$

$$= f(x, \underbrace{y_n + h \left.\frac{dy}{dx}\right|_{x_n}}_{\text{small}})$$

Taylor expand again the function f

$$f(x, y(n)) \cong f(x, y_n) + h \underbrace{\boxed{\left.\frac{dy}{dx}\right|_{x_n}}}_{\substack{\text{slope at } x_n \\ f(x_n, y_n)}} f'(x, y_n)$$

$$\int_{x_n}^{x_{n+1}} f(x, y(x))\,dx \cong \int_{x_n}^{x_{n+1}} \left[f(x, y_n) + h f(x_n, y_n) f'(x, y_n) \right] dx$$

fix the function values

fix the function values at our initial condition at x_n

$$\cong h \cdot f(x_n, y_n) + \underbrace{h^2 f(x_n, y_n) f'(x_n, y_n)}$$

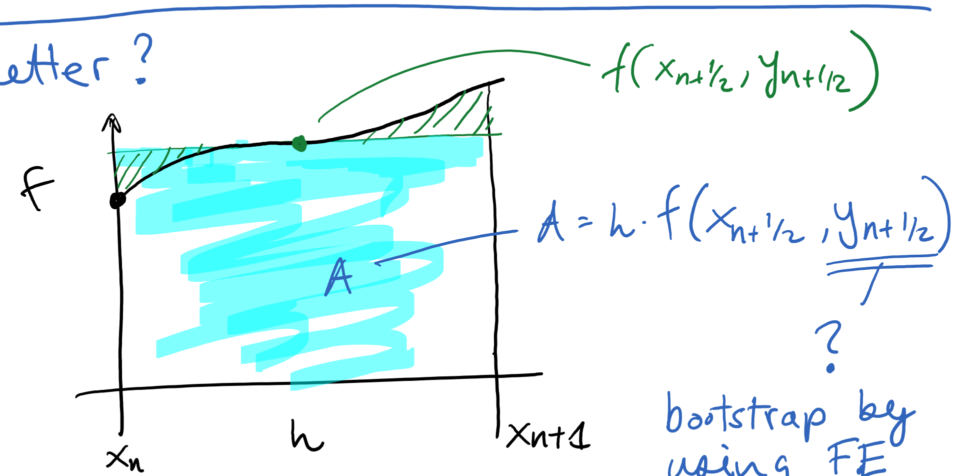
Local Error: $\mathcal{O}(h^2)$

Global error: $N_{steps} = \frac{X}{h}$

Global error $\Rightarrow \mathcal{O}(h)$

Quite Poor.

Can we do better?



bootstrap by using FE method for 1/2 step.

$$y_{n+1/2} = y_n + \frac{h}{2} f(x_n, y_n)$$

use this in the integral

$$y_{n+1} - y_n = h \cdot \underline{f}\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} \underline{f}(x_n, y_n)\right)$$

2 evaluations of f here (so more expensive)

Local Error: $\mathcal{O}(h^3)$

Global Error: $\mathcal{O}(h^2)$

Midpoint Runge-Kutta:

$$y_{n+1} = y_n + h \cdot f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

---

higher order Integration → Simpson's Rule

4^{th} order R-K

$$\underline{k_1} = h \cdot \underline{f}(x_n, y_n)$$
$$\underline{k_2} = h \cdot \underline{f}\left(x_n + \frac{h}{2}, \underline{y}_n + \frac{\underline{k_1}}{2}\right)$$
$$\underline{k_3} = h \cdot \underline{f}\left(x_n + \frac{h}{2}, \underline{y}_n + \frac{\underline{k_2}}{2}\right)$$
$$\underline{k_4} = h \cdot \underline{f}\left(x_n + h, \underline{y}_n + \underline{k_3}\right)$$

$$y_{n+1} = y_n + \frac{\underline{k_1}}{6} + \frac{\underline{k_2}}{3} + \frac{\underline{k_3}}{3} + \frac{\underline{k_4}}{6} + \mathcal{O}(h^5)$$

Explicit Methods

Predator-Prey Behaviour

Foxes and Mice

f $\quad$ m

Lotka-Volterra Model (1920)

without foxes we want the mice population to grow exponentially

$$\frac{\Delta m}{m} = k_m \cdot \Delta t$$

↑ constant birth rate

but if foxes are around the population reduces proportional to the number of foxes.

$$\frac{\Delta m}{m} = k_m \cdot \Delta t - k_{mf} \cdot f \cdot \Delta t$$

$$\frac{dm}{dt} \approx \frac{\Delta m}{\Delta t} = (k_m \cdot m - \underbrace{k_{mf} \cdot m \cdot f}_{\substack{\propto \text{ number of} \\ \text{encounters of} \\ \text{foxes and mice}}})$$

$$\frac{\Delta f}{f} = -k_f \Delta t$$

k_f ← exponential death rate for foxes.

$$\frac{\Delta f}{f} = -k_f \Delta t + k_{fm} m \cdot \Delta t$$

k_{fm} – Birthrate $\propto$ to number of mice

$$\boxed{\begin{aligned} \frac{df}{dt} &= -k_f \cdot f + k_{fm} \cdot f \cdot m \\ \frac{dm}{dt} &= k_m m - k_{mf} \cdot f \cdot m \end{aligned}}$$

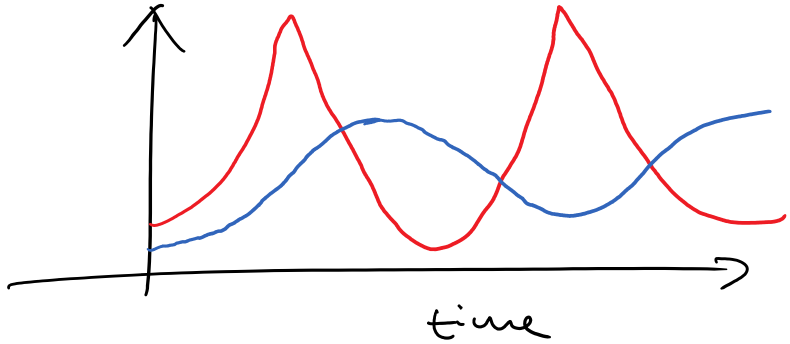
Solve this for some initial populations.

$k_m = 2$
$k_{mf} = 0.02$
$k_{fm} = 0.01$
$k_f = 1.06$

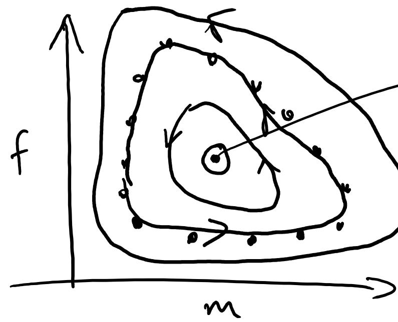
I.C. $m(0) = 100$
$f(0) = 15$

$\underline{y} = \langle m(t), f(t) \rangle$

2 Plots:



? f, m



$$\left.\begin{array}{l}\dfrac{df}{dt}=0\\[2ex] \dfrac{dm}{dt}=0\end{array}\right\}\ \text{fixed point}$$

"Phase" Plot