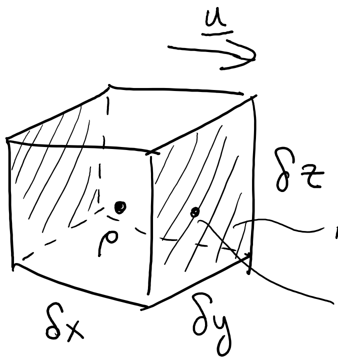
SINS 11: Hyperbolic PDEs

25 November 2019 13:01



Area $\delta y \delta z$

$$\rho(x+\tfrac{1}{2}\delta x, t) \approx \rho(x,t) + \tfrac{1}{2}\delta x \left.\frac{\partial \rho}{\partial x}\right|_{x,t}$$

§

Flux = material flowing through the surface per unit time.

$$\rho \Delta x \cdot \delta y \delta z / \Delta t = \underbrace{\rho \underline{u}}\cdot \delta x \delta y$$

$\rho \underline{u}$ at $x+\delta x$ is $\approx \rho \underline{u} + \frac{1}{2}\delta x \left.\frac{\partial(\rho \underline{u})}{\partial x}\right|_{\underline{x},t}$

$$\left\{ \cancel{\rho \underline{u}} + \tfrac{1}{2}\delta x \left.\frac{\partial(\rho u)}{\partial x}\right|_{\underline{x},t} \right\} \delta y \delta z$$

Flux through the "front" surface

$$\left\{ \cancel{\rho \underline{u}} - \tfrac{1}{2}\delta x \left.\frac{\partial(\rho u)}{\partial x}\right|_{\underline{x},t} \right\} \delta y \; \delta z$$

to simplify let $\underline{u}$ in x direction only.

Flux <u>into</u> the cube is then given by the difference of the above:

$$-1\, \delta x \left.\frac{\partial(\rho \underline{u})}{\partial x}\right|_{\underline{x},t} \delta y \delta z = -\frac{\partial(\rho \underline{u})}{\partial x}\delta V$$

$$\boxed{-\frac{\partial(\rho \underline{u})}{\partial x}\cancel{\delta V} = \frac{\partial \rho}{\partial t}\cdot \cancel{\delta V}}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho \underline{u})}{\partial x}$$ for flow in x direction.

$$\frac{\partial \rho}{\partial t} = -\nabla\cdot(\rho \underline{u}) \overset{\text{or}}{\Rightarrow} \boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u}) = 0}$$

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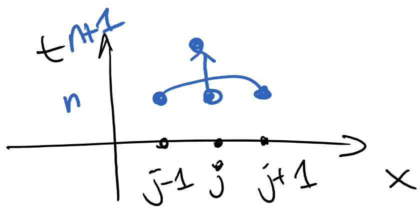
Conservation of Mass.

+ Conservation of Momentum (3 equations)

+ Conservation of Energy (1 equation)

for later ...

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0$$ "Linear Advection Equation"



$$\left.\frac{\partial \rho}{\partial x}\right|_j \approx \frac{\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)}}{2\Delta x}$$

$$\frac{\partial \rho}{\partial t} \approx \frac{\rho_j^{(n+1)} - \rho_j^{(n)}}{\Delta t}$$

$$\rho_j^{(n+1)} = \rho_j^{(n)} - \frac{1}{2}c\left(\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)}\right)$$

Courant Number $c \equiv \frac{\Delta t\, u}{\Delta x}$

von Neumann Stability Analysis

$$\rho_j^{(n)} = A^n e^{ij\theta} \qquad i \equiv \sqrt{-1}$$

$$\cancel{A^{n+1}} \cancel{e^{ij\theta}} = \cancel{A^n} \cancel{e^{ij\theta}} - \frac{1}{2}c\cancel{A^n}\left(e^{i(j+1)\theta} - e^{i(j-1)\theta}\right)$$

cancel $\cancel{e^{ij\theta}}$

$$A = 1 - c\left(\frac{e^{i\theta} - e^{-i\theta}}{2}\right)$$

$$\boxed{A = 1 - ic\sin(\theta)}$$

$$A^*A = ||A|| = 1 + c^2\sin^2\theta > 1!!$$

Completely useless

Completely useless

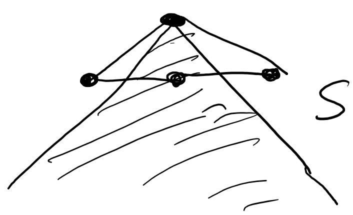
LAX Method: $\rho_j^{(n)} \longrightarrow \frac{1}{2}\left(\rho_{j+1}^{(n)} + \rho_{j-1}^{(n)}\right)$

$$\rho_j^{(n+1)} = \frac{1}{2}\left(\rho_{j+1}^{(n)} + \rho_{j-1}^{(n)}\right) + \frac{1}{2} c \left(\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)}\right)$$

$\Longrightarrow$ $A = \cos\theta - i c \sin\theta$

Stable when $|c| \leq 1$ or $\frac{|a| \Delta t}{\Delta x} < 1$

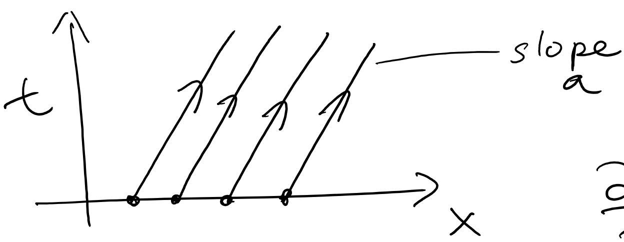
Courant Condition
or CFL Condition

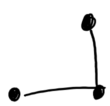


$$a < \frac{\Delta x}{\Delta t}$$

$$\frac{a \Delta t}{\Delta x} < 1$$

Information <u>must</u> be able to flow on the grid faster than in "reality".





$\frac{\partial \rho}{\partial x} \approx \frac{\rho_j^{(n)} - \rho_{j-1}^{(n)}}{\Delta x}$ when $a > 0$

$\frac{\rho_{j+1}^{(n)} - \rho_j^{(n)}}{\Delta x}$ when $a < 0$

CFL: $\|A\| < 1$ when

$$0 \leq c < 1$$

Upwind Schemes

$$\rho_j^{(n+1)} = \rho_j^{(n)} - c\left(\rho_j^{(n)} - \rho_{j-1}^{(n)}\right) \text{ for } a > 0$$

LAX-WENDROFF Method:

$$\rho_j^{(n+1)} = \frac{1}{2}c(1+c)\rho_{j-1}^{(n)} + (1-c^2)\rho_j^{(n)} - \frac{1}{2}c(1-c)\rho_{j+1}^{(n)}$$

for c being +ve or -ve!