QM Introduction

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In classical mechanics the state of a system is a point in phase space e.g. (x, p). That is possible because x and p can be measured simultaneously.

In quantum mechanics this is no longer true since the state of the system gets corrupted by observation. We need a completely new description.

Postulates

System state: Vector in a vector space $|\varphi\rangle$

Observable: Hermitian operator $O = O^\dagger$

Expectation value of O: $\langle\varphi|O|\varphi\rangle = o \in \mathbb{R}$

where $O|\varphi\rangle = o|\varphi\rangle$, $|\varphi\rangle$ is eigenstate of O and o eigenvalue.

Probability of finding system in state $|\varphi\rangle$: $\|\varphi\| = \sqrt{\langle\varphi|\varphi\rangle}$

For $|\varphi\rangle = (\ldots, \varphi_i, \ldots)$: $\langle\varphi|\psi\rangle = \sum_i \varphi_i^* \psi_i$

Time evolution:

$$|\varphi(t+\varepsilon)\rangle = \left(1 - i\frac{\varepsilon}{\hbar}H\right)|\varphi(t)\rangle \Rightarrow \frac{|\varphi(t+\varepsilon)\rangle - |\varphi(t)\rangle}{\varepsilon} = -\frac{i}{\hbar}H|\varphi(t)\rangle$$

For infinitesimal ε: $i\hbar\frac{\partial}{\partial t}|\varphi(t)\rangle = H|\varphi(t)\rangle$ Schrödinger Equation

Formal solution

$$|\varphi(t)\rangle = e^{-\frac{i}{\hbar}tH}|\varphi(0)\rangle$$

Possible unitary approximation ($\hbar = 1$):

$$e^{-itH} \cong \frac{1 - i\frac{t}{2}H}{1 + i\frac{t}{2}H} + O(t^3)$$

$\Rightarrow$ Implicit scheme

Claim: $e^{2x} \cong \dfrac{1+x}{1-x} + O(x^3)$

Proof:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots \qquad |x| < 1$$

$$\begin{aligned}\frac{1+x}{1-x} &= (1+x)(1 + x + x^2 + x^3 + \ldots) \\ &= 1 + x + x^2 + x^3 + \ldots \\ &\quad + x + x^2 + x^3 + \ldots\end{aligned}$$

$\Rightarrow$ Implicit scheme

$$= 1 + x + x^2 + x^3 + \dots$$
$$+ x + x^2 + x^3 + \dots$$
$$= 1 + 2x + 2x^2 + 2x^3 + \dots$$
$$= 1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{4} + \dots$$

$$e^{2x} = 1 + 2x + (2x)^2$$

$$\Rightarrow \text{Error} = \left(\frac{1}{4} - \frac{1}{6}\right)(2x)^3 = \frac{8}{12}x^3$$
(leading order)

Creation and anihilation operators for fermions

Only $\boxed{\text{one}}$ particle can occupy a certain state $\Rightarrow$ Only two states $|0\rangle$ (no particle) and $|1\rangle$ (one particle)

c^+ creates a particle, c anihilates a particle

$$c^+|0\rangle = |1\rangle \qquad c|0\rangle = 0$$
$$c^+|1\rangle = 0 \qquad c|1\rangle = |0\rangle$$

$$cc^+(|0\rangle + |1\rangle) = c(|1\rangle + 0) = |0\rangle$$
$$c^+c(|0\rangle + |1\rangle) = c^+(0 + |0\rangle) = |1\rangle$$
$$(cc^+ + c^+c)(|0\rangle + |1\rangle) = |0\rangle + |1\rangle$$

$$\Rightarrow cc^+ + c^+c = 1$$
def
$\{c, c^+\}$ Anti-commutator

$$\{c, c\} = 0$$
$$\{c^+, c^+\} = 0$$

More general
$$\{c_\ell, c_k^+\} = \delta_{\ell k}$$
$$\{c_\ell, c_k\} = 0$$
$$\{c_\ell^+, c_k^+\} = 0$$

In the lecture we had expressions like

$$(c_\ell^+ c_{\ell'} + c_{\ell'}^+ c_\ell)\, c_\ell^+|0\rangle$$

The "trick" is to pull through the c's until they act on the vacuum.

The result is the terms comming from $\{c_\ell, c_k^+\} = \delta_{\ell k}$

$$c_\ell^+ c_{\ell'} c_\ell^+ |0\rangle = c_\ell^+ c_\ell^+ c_{\ell'} |0\rangle = 0$$

$$c_{\ell'}^+ c_\ell c_\ell^+ |0\rangle = c_{\ell'}^+ (1 - c_\ell^+ c_\ell) |0\rangle = c_{\ell'}^+ |0\rangle$$

$$\left. \begin{matrix} \\ \\ \end{matrix} \right\} \left(c_\ell^+ c_{\ell'} + c_{\ell'}^+ c_\ell\right) c_\ell^+ |0\rangle = c_{\ell'}^+ |0\rangle$$

The operator $c_\ell^+ c_{\ell'} + c_{\ell'}^+ c_\ell$ moves a particle from site ℓ to site ℓ'

Applying the operator twice: $\ell \to \ell' \to \ell \Rightarrow$ Unit operator

$\Rightarrow$ Even powers $\sim$ Unit operator $\Rightarrow$ Terms of series repr. for cos

Odd powers $\sim$ Single operator $\Rightarrow$ " " " " " sin