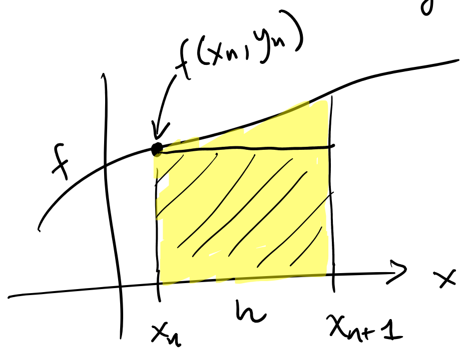
Ordinary Differential Equations

Monday, 16 October 2023 12:56

$$\frac{dy(x)}{dx} = \underline{f}(x, y) \qquad \int_{y_n}^{y_{n+1}} dy = \int_{x_n}^{x_{n+1}} \underline{f}(x,y)\, dx$$



$\downarrow$ approximate with the ▨ rectangle

$$y_{n+1} - y_n = f(x_n, y_n) \cdot h$$

$$\boxed{y_{n+1} = y_n + h \cdot f(x_n, y_n)}$$

Forward Euler Method

$$y(x) \approx y_0 \to y_1 \to y_2 \to y_3 \cdots \to y_{n+1}$$

Error depends on the stepsize h.

Truncation error: error associated with the algorithm or method, and not the precision of the floating point calculations (roundoff error).

$$y_n = \underbrace{0.00134}_{\mathcal{O}(h^p)} \cdots 3\underbrace{67}_{\mathcal{O}(\sqrt{N})}$$

Roundoff Error

Error of Forward Euler:

$$\int_{x_n}^{x_{n+1}} f(x,y)\, dx \qquad f(x, y(x)) = f(x, \overbrace{y(x_n + h)}^{\text{Taylor Expansion}})$$

$$= \underbrace{f\Big(x, y_n + \underbrace{h \cdot \frac{dy}{dx}\Big|_{x_n}}_{\text{small}}\Big)}_{\text{Taylor Expand a second time}}$$

derivative of second argument

$$f(x, y(x)) \cong f(x, y_n) + h \cdot \underbrace{\frac{dy}{dx}\Big|_{x_n}}_{f(x_n, y_n)} \cdot f'(x, y_n)$$

$$\int_{x_n}^{x_{n+1}} f(x, y(x))\, dx \cong \int_{x_n}^{x_{n+1}} \left[f(x, y_n) + h\, f(x_n, y_n)\, f'(x, y_n)\right] dx$$

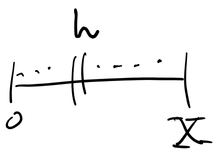
$$\int_{x_n}^{x_{n+1}} f(x,y(x))\,dx \cong \int_{x_n}^{x_{n+1}} \left[f(x,y_n) + h\, f(x_n,y_n)\, f'(x,y_n) \right] dx$$

$f(x_n, y_n)$

fix the function values at our initial condition at x_n

$$y_{n+1} - y_n \cong h\cdot f(x_n,y_n) + \underbrace{\boxed{h^2} f(x_n,y_n)\cdot f'(x_n,y_n)}_{O(h^2)}$$

Single step error "Local" Error



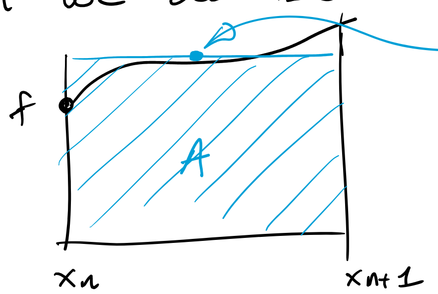
$$N_{steps} = \frac{X}{h}$$

Global Error over the interval X

$$N_{steps}\cdot O(h^2) = O(h)$$

Very Poor

Can we do better?



$f(x_{n+\frac{1}{2}}, y_{n+\frac{1}{2}})$

$$A = h\cdot f(x_{n+\frac{1}{2}}, \underline{y_{n+\frac{1}{2}}})$$

?

Predict it

predictor: $y_{n+\frac{1}{2}} = \underline{y_n + \frac{h}{2} f(x_n, y_n)}$

corrector: $y_{n+1} = y_n + h\cdot \boxed{f}\left(x_n + \frac{h}{2},\, y_n + \frac{h}{2} \boxed{f}(x_n, y_n)\right)$

2 evaluations of the right hand side function.

Local Error : $O(h^3)$

Gobal Error : $O(h^2)$

Higher order method $\rightarrow$ price is more evaluations of f.

Midpoint Runge-Kutta

Most popular is 4th order method

Most popular is 4th order method
(based on using Simplex method for calculating the area)

$$\underline{k_1} = h \cdot \underline{f}(x_n, y_n)$$
$$\underline{k_2} = h \cdot \underline{f}(x_n + \tfrac{h}{2}, y_n + \tfrac{\underline{k_1}}{2})$$
$$\underline{k_3} = h \cdot \underline{f}(x_n + \tfrac{h}{2}, y_n + \tfrac{\underline{k_2}}{2})$$
$$\underline{k_4} = h \cdot \underline{f}(x_n + h, y_n + \underline{k_3})$$
$$y_{n+1} = y_n + \frac{\underline{k_1}}{6} + \frac{\underline{k_2}}{3} + \frac{\underline{k_3}}{3} + \frac{\underline{k_4}}{6} + O(h^5)$$

Stable? <u>Explicit</u> / Implicit

Predator - Prey Behaviour

Foxes and Mice

f m

Lotka-Volterra Model (1920)

without foxes ($f=0$) we want the mice (m) to grow exponentially

$$\frac{\Delta m}{m} = k_m \cdot \Delta t$$

constant birth rate

but if foxes are around the population reduces proportional to the number of foxes

$$\frac{\Delta m}{m} = k_m \cdot \Delta t - k_{mf} \cdot f \Delta t$$

$$\frac{\Delta m}{\Delta t} = k_m \cdot m - k_{mf} \cdot \underbrace{m \cdot f}_{\propto \text{ number of encounters}}$$

$$\frac{dm}{dt} = (k_m \cdot m - k_{mf} \cdot m \cdot f)$$

$$\frac{dm}{dt} = (k_m \cdot m - k_{mf} \cdot m \cdot f)$$

for the foxes:

$$\frac{\Delta f}{f} = -k_f \Delta t$$

exponential death rate

$$\boxed{\begin{aligned} \frac{df}{dt} &= -k_f \cdot f + k_{fm} \cdot f \cdot m \\ \frac{dm}{dt} &= k_m \cdot m - k_{mf} \cdot m \cdot f \end{aligned}}$$

Solve this for a given initial population

$k_m = 2$

$k_{mf} = 0.02$

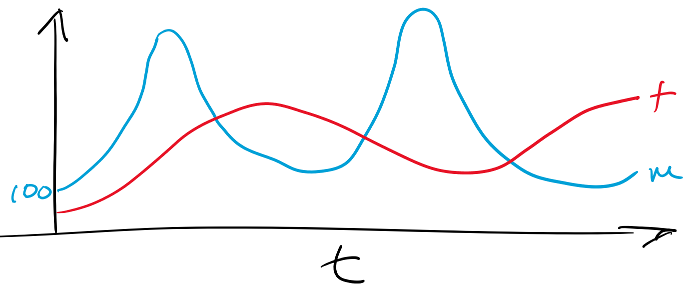
$k_{fm} = 0.01$

$k_f = 1.06$

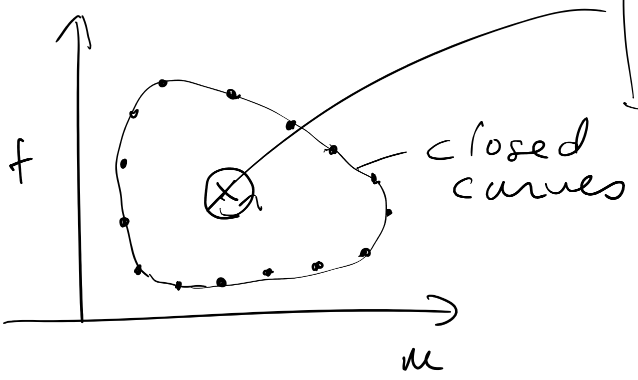
IC $\quad m(0) = 100$

$f(0) = 15$

$y = \langle m(t), f(t) \rangle$



1. Use F-E
2. Use midpoint RK2
3. $\boxed{\text{use RK4}}$



$$\boxed{\frac{df}{dt} = 0, \quad \frac{dm}{dt} = 0}$$

FIXED POINT