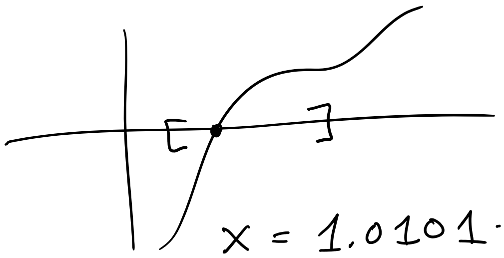
Newton's Method and solving Kepler's Equation

Monday, 25 September 2023 12:53

$f(x) = 0$

$|a-b| < \varepsilon_{absolute}$

$\frac{|a-b|}{|c|} < \varepsilon_{relative}$



$x = 1.0101\ldots$

$\longrightarrow$ one digit with each iteration

$$\boxed{\begin{array}{l}\text{float (32 bit) mantissa} \Rightarrow 23 \text{ iterations} \\ \text{double (64 bit) mantissa} \Rightarrow 52 \text{ iterations}\end{array}}$$

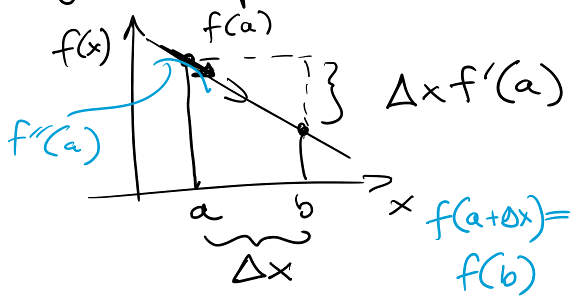
Rate of convergence $\longrightarrow$ linear convergence
constant number of bits/digits per iteration.

Quadratic Convergence
Number of digits/bits double per iteration.

Newton's Method

Taylor Expansion

$$f'(a) \equiv \left.\frac{d\,f(x)}{dx}\right|_a$$



$$f(b) \approx f(a) + \Delta x f'(a)$$

$$\underbrace{f(a+\Delta x)}_{f(b)} = f(a) + \Delta x \cdot f'(a) + \frac{\Delta^2}{2} \cdot f''(a)$$

$$f(b) \equiv f(a+\Delta x) \cong f(a) + \Delta x \cdot f'(a) + \cdots + \frac{1}{n!}\Delta x^n F^{(n)}(a)$$

$$+ \left[\frac{\Delta x^n}{(n-1)!}\int_0^1 (1+t)^{n-1} f^{(n)}(a+t\Delta x)\, dt\right]$$

$$f(x) = f(a+\Delta x) \cong f(a) + \Delta x\, f'(a)$$

Let's suppose this function has a root at x. $f(x) = 0$

$$f(a) + \Delta x\, f'(a) \cong 0$$

$$\Delta x \cong -\frac{f(a)}{f'(a)}$$

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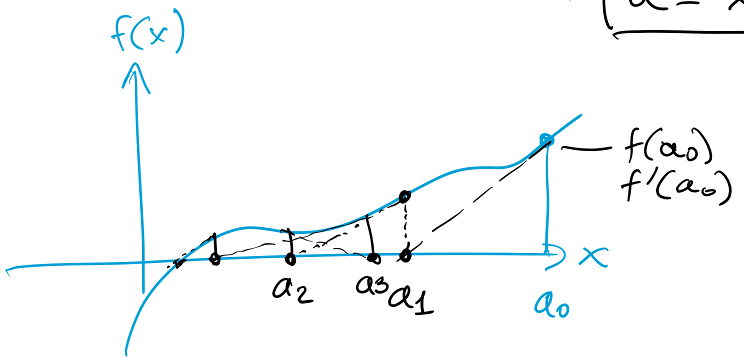
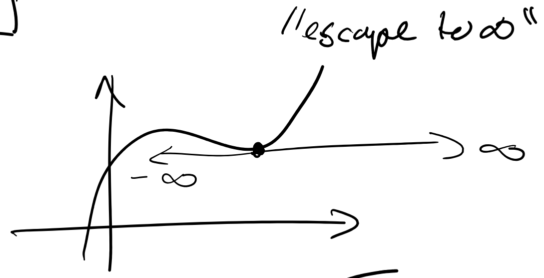
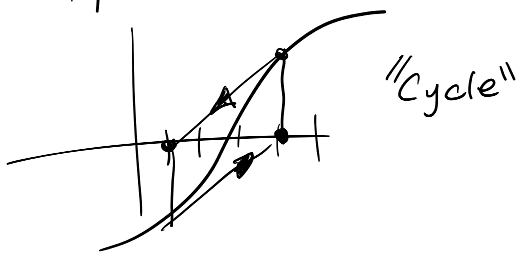
Newton's Method

$$\boxed{x = a + \Delta x \cong a - \frac{f(a)}{f'(a)}}$$

$$\boxed{a = x}$$







When its working it is really fast!

$f'(a)$?

In computer $\oplus$ $\circledast$ $\sqrt{x}$, a/b, $\sin x$?

$$\sqrt{x} \longrightarrow \left(\frac{1}{\sqrt{x}}\right) \cdot x \qquad \frac{a}{b} \longrightarrow a \cdot \left(\frac{1}{\sqrt{b}}\right)\left(\frac{1}{\sqrt{b}}\right)$$

$y = \frac{1}{\sqrt{x}}$ y is the unknown and x is given

$$0 = f(y) = y - \frac{1}{\sqrt{x}}$$

This is not so great $\quad y^2 = \frac{1}{x}$

$f(y) = x - \frac{1}{y^2}$ weird, but it works!

$$y \leftarrow y - \frac{x - \frac{1}{y^2}}{2\frac{1}{y^3}}$$

$$y \leftarrow y - \frac{1}{2} x y^3 + \frac{1}{2} y$$

$$y \leftarrow y * (1.5 - 0.5 * x * y * y)$$

$$x = \pm M \times 2^e \qquad 1 \le M < 2$$

$$x = +M \times 2^{e} \qquad 1 \le M < 2$$

$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{M}} \times 2^{-e/2} \; \checkmark$$

$$\frac{\sqrt{2}}{2} < y < 1 \qquad \approx 0.8535$$

Today's Assignment: Solve Kepler's Equation

$$\underline{\underline{M}} = E - \underline{\underline{e}} \sin E$$

given

solve!

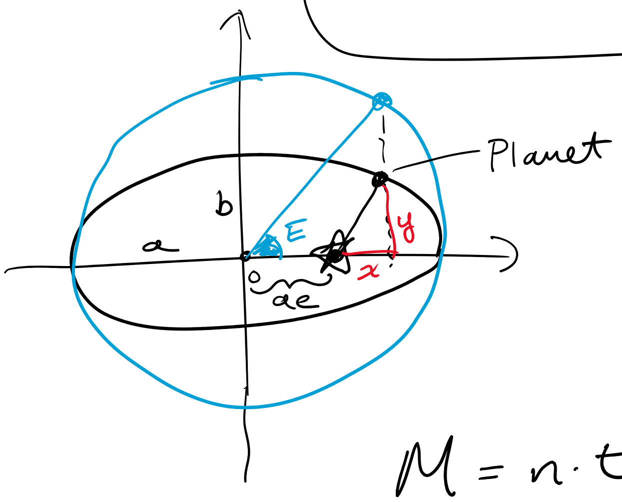
$$E(M,e) = ?$$

$$f(E) = 0 \, !$$

$$0 = E - e\sin E - M = f(E)$$

$$? = f'(E)$$

$$E_{new} = E_{OLD} - \frac{f(E_{OLD})}{f'(E_{OLD})}$$



$$x = a \cdot \cos E - ae$$

$$y = b \cdot \sin E$$

$$y = a\sqrt{1-e^2}\,\sin E$$

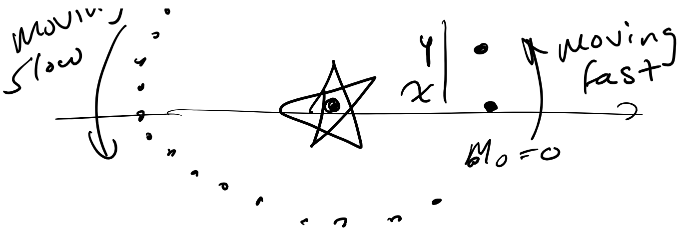
$$M = n \cdot t \qquad n = \frac{2\pi}{T} \text{ - period in years}$$

$$T = a^{3/2}$$

AU

$e = 0.5 \quad a = 1$

$\Delta t = 1/50$



Initial $E_0 = M$