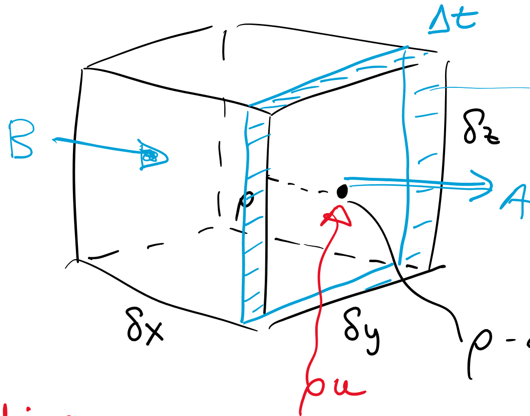
Very small cube



$\frac{\partial \rho}{\partial t}$

blue material is $\rho\, \Delta x\, \delta y\, \delta z$

Material leaving per unit time is $\underbrace{\left[\underbrace{\rho \frac{\Delta x}{\Delta t}}_{\rho u} \right]}_{\text{Flux}} \delta y\, \delta z$

ρ - density at the wall

Define
$\underline{v} := \langle u_x, u_y, u_z \rangle$

$$\rho(x + \tfrac{1}{2}\delta x, t) \approx \rho(x,t) + \frac{1}{2}\delta x \left.\frac{\partial \rho}{\partial x}\right|_{x,t}$$

Flux through the surface is given by

$$\rho u(x + \tfrac{1}{2}\delta x, t) \approx \rho u(x,t) + \frac{1}{2}\delta x \left.\frac{\partial(\rho u)}{\partial x}\right|_{x,t}$$

A: $\left\{\rho u + \frac{1}{2}\delta x \frac{\partial(\rho u)}{\partial x}\right\} \delta y\, \delta z$ is the material leaving the cube.

Flux through the opposite face is then,

B: $\left\{\rho u - \frac{1}{2}\delta x \frac{\partial(\rho u)}{\partial x}\right\} \delta y\, \delta z$
Material entering the cube

$$\frac{\partial \rho}{\partial t} \cdot \cancel{\delta x \delta y \delta z} = \underbrace{-1\, \cancel{\delta x} \frac{\partial(\rho u)}{\partial x} \cancel{\delta y \delta z}}_{= B - A}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho u_x)}{\partial x}$$ if there is only velocity in the x-direction
$u = u_x$!

Adding the other faces:

$$-\frac{\partial(\rho u_y)}{\partial y}$$
$$-\frac{\partial(\rho u_z)}{\partial z}$$

$$\nabla \cdot (\underline{A}) := \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= -\nabla \cdot (\rho \underline{u})$$

$$\equiv -\nabla\cdot(\rho \underline{u})$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \underline{u}) = 0}$$

Hyperbolic P.D.E.

Based on one principle:
conservation of the "material" whose density is given by ρ.

If we have
① conservation of Mass
② conservation of Energy
③ conservation of Momentum

☆ Gives the Equations of Hydrodynamics (without viscosity) Euler Equations

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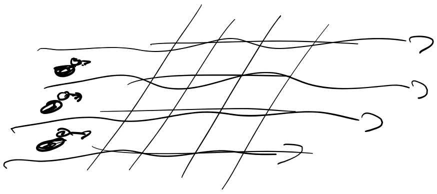
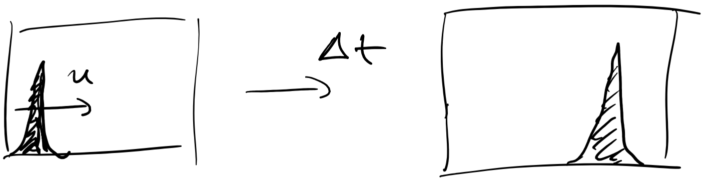
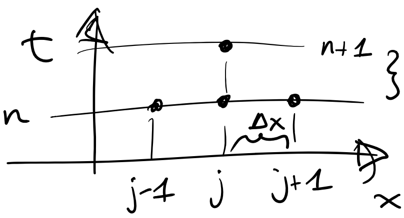
Linear Advection Equation



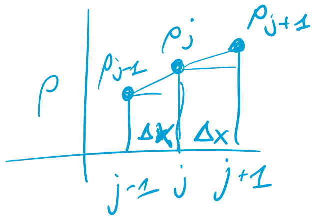


$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0$$



$$\left.\frac{\partial \rho}{\partial t}\right|_j \approx \frac{\rho_j^{n+1} - \rho_j^{n}}{\Delta t}$$

$$\left.\frac{\partial \rho}{\partial x}\right|_j^n \approx \frac{\rho_{j+1}^{n} - \rho_{j-1}^{n}}{2\Delta x}$$



$$\rho_j^{(n+1)} = \rho_j^{(n)} - \frac{1}{2}C\left(\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)}\right)$$

$$C = \Delta t \cdot u$$

$$C \equiv \frac{\Delta t \cdot u}{\Delta x}$$

Von Neumann Stability Analysis

$$\rho_j^{(n)} = A^n e^{ij\theta} \qquad i = \sqrt{-1}$$

$$A^{n+1} e^{ij\theta} = A^n e^{ij\theta} - \tfrac{1}{2} C A^n \left(e^{i(j+1)\theta} - e^{i(j-1)\theta}\right)$$

cancel $A^n e^{ij\theta}$ everywhere

$$A = 1 - C \underbrace{\left(\frac{e^{i\theta} - e^{-i\theta}}{2}\right)}_{i \sin\theta}$$

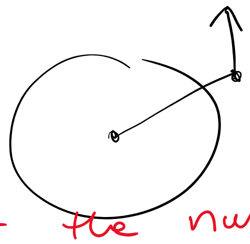
$$\boxed{A = 1 - iC\sin\theta}$$

$$A^* A = |A|^2 = 1 + C^2 \sin^2\theta$$

No matter what the values of C and θ are

This $|A|^2 > 1$

Recall $\rho \propto A^n$



Solution diverges.

Disaster: Means that the numerical method is unstable.

<u>Method is useless!</u>

Simple Modification (LAX method):

Replace $\rho_j^{(n)} \longrightarrow \frac{1}{2}\left(\rho_{j+1}^{(n)} + \rho_{j-1}^{(n)}\right)$

replace with average of the 2 neighbor points.

$(n+1)$

'the 2 neighbor points.

$$\rho_j^{(n+1)} = \frac{1}{2}\left(\rho_{j+1}^{(n)} + \rho_{j-1}^{(n)}\right) - \frac{1}{2}c\left(\rho_{j+1}^{(n)} - \rho_{j-1}^{(n)}\right)$$

Same von Neumann Stability Analysis:

$$A = \cos\theta - i c \sin\theta$$

$$|A|^2 = \cos^2\theta + c\sin^2\theta \quad > 1 \,?$$

It is Stable if $|c| < 1$!

Courant condition

$$|u| < \frac{\Delta x}{\Delta t}$$

Physical velocity

Grid velocity

"Fastest information can flow on the grid."

Method Upwind Methods:

$$\rho_j^{(n+1)} = \rho_j^{(n)} - c\left(\rho_j^{(n)} - \rho_{j-1}^{(n)}\right) \quad \text{for } u > 0$$

$$\qquad\qquad - c\left(\rho_{j+1}^{(n)} - \rho_j^{(n)}\right) \quad \text{for } u < 0$$

First order upwind method

$$\boxed{|c| < 1}$$

LAX-Wendroff method (upwind)

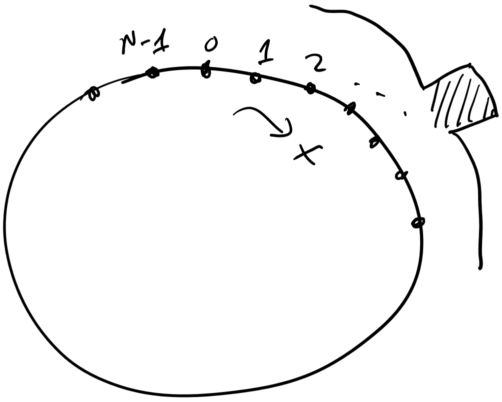
$$\rho_j^{(n+1)} = \frac{1}{2}c(1+c)\rho_{j-1}^{(n)} + (1-c^2)\rho_j^{(n)} - \frac{1}{2}c(1-c)\rho_{j+1}^{(n)}$$

for $u > 0$

Periodic B.C.s

0. Loser method
1. LAX

Periodic B.C.s



0. Loser method

1. LAX

2. 1^{ST}-order Upwind

3. Lax-Wendroff