4. Measurement of the modulus of elasticity

4.1 Introduction

To name a few examples, a precise understanding of the mechanical properties of a given material is essential in the calculation of bridge constructions or bearing architectural elements. In those cases, one can not regard the body as being rigid but has to consider a plasticity due to external forces (also see lab course TB from the winter semester). An important quantity in this particular context is the modulus of elasticity, which specifies the relative linear expansion of a material for a given tensile stress.

In this experiment, one shall determine the modulus of elasticity for aluminum, steel and brass by measuring the speed of sound inside these materials.

4.2 Theory

a) Definition of the modulus of elasticity

Arranging a rod of length l, as shown in Fig. 4.1, and applying a horizontal force $\vec{F}$, it will undergo an expansion in length Δl due to this force.

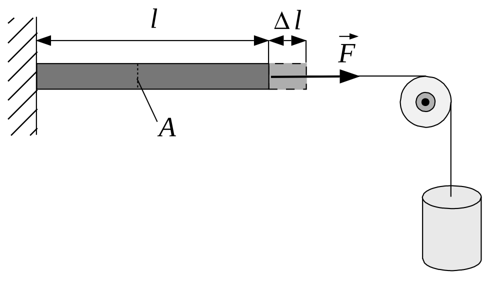

Figure 4.1: Concerning the definition of the modulus of elasticity.

In case of the force acting evenly over the whole cross sectional area of the rod, it creates a tensile stress of $\sigma = F/A$. For sufficiently small tensions, the rods relative expansion in length $\varepsilon = \Delta l/l$ is proportional to the acting force. This is the regime of the so-called Hooke's law

$$\sigma = E \cdot \varepsilon \tag{4.1}$$

The constant of proportionality, E, is a material property and is being denoted as the modulus of elasticity.

In principle, the modulus of elasticity for a given material can be determined directly via Equ. 4.1 by measuring the relative expansion in length as a function of the adjacent tensile stress in the range of proportionality. Since the emerging expansions are very small in general, a precise measurement is not possible using simple techniques and devices. It is easier to determine the modulus of elasticity by measuring the speed of sound inside a thin rod.

b) Correlation of the modulus of elasticity and the speed of sound

After striking a thin rod perpendicularly to its cross sectional area, the brought about disturbance of the material propagates along the rod in form of an acoustic wave. The passing wave temporarily dislocates the concerned cross sectional areas of the rod from their resting positions. This causes shearing stresses inside of the material. The displacements u of the cross sectional areas and the resulting shearing stresses σ are functions of time t and position x along the rod (see Fig. 4.2).

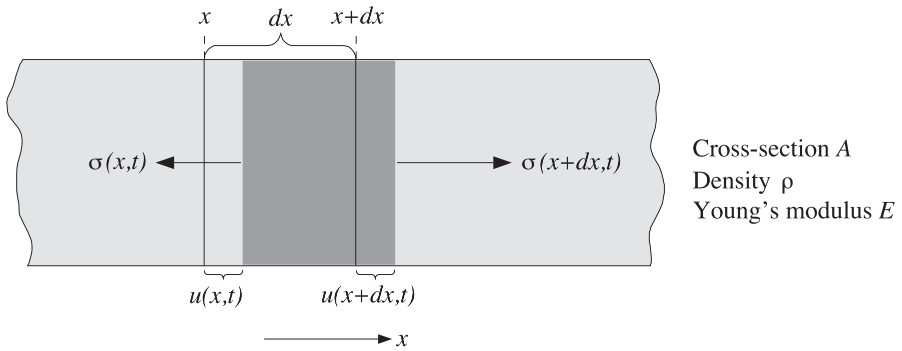

Figure 4.2: Concerning the propagation of a longitudinal acoustic wave inside of a rod.

At a given time t, a force

$$F = (\sigma(x+dx) - \sigma(x)) \cdot A \tag{4.2}$$

acts on a volume element with a cross sectional area A, thickness dx and position x.

Let ρ be the density of the material, then $m = \rho \cdot dV = \rho \cdot A \cdot dx$ is the mass of the concerned volume element. According to Newtons 2nd law one has

$$\rho \cdot A \cdot dx \cdot \frac{d^2u}{dt^2} = (\sigma(x+dx) - \sigma(x)) \cdot A \tag{4.3}$$

and therefore

$$\rho \cdot \frac{d^2u}{dt^2} = \frac{\sigma(x+dx) - \sigma(x)}{dx} = \frac{d\sigma}{dx} \tag{4.4}$$

On the other hand, the volume element undergoes a relative length contraction

$$\varepsilon = \frac{\Delta(dx)}{dx} = \frac{u(x+dx)-u(x)}{dx} = \frac{du}{dx} \tag{4.5}$$

and by applying Hooke's law one has

$$\sigma = E \cdot \varepsilon = E \cdot \frac{du}{dx} \tag{4.6}$$

By substituting Equ. 4.6 into Equ. 4.4, it follows that

$$\frac{d^2u}{dt^2} = \frac{E}{\rho} \cdot \frac{d^2u}{dx^2} \tag{4.7}$$

This is the differential equation of a plane wave with propagation speed

$$v = \sqrt{\frac{E}{\rho}} \tag{4.8}$$

In case of a known density of the material, one can determine its modulus of elasticity by measuring the speed of sound inside the rod.

c) Determination of the speed of sound out of the frequency and the wavelength

A determination of the wavelength by measuring the duration it takes the sound wave to travel a certain distance in the rod is not possible. In the problem in hand, the distance is limited by the length of the rod and therefore the runtimes to be measured would be extremely short. Therefore, in this experiment, the speed of sound is being determined via the relation

$$v = \lambda \cdot \nu \tag{4.9}$$

out of the wavelength λ and the frequency ν from a standing longitudinal wave in the rod.

For this purpose, the rod is being chucked firmly in its center, such that the two ends can oscillate freely. Rubbing the rod lets it oscillate longitudinally. Due to the reflections of running waves at the open endings of the rod and the superposition of primal and reflected waves, standing waves arise in the rod. Consider a wave which originally runs in positive x-direction,

$$u_1(x,t) = u_0 \cdot \cos\left\{k(x - vt)\right\} \qquad\qquad k = \frac{2\pi}{\lambda} \tag{4.10}$$

and the reverse, reflected waves with same amplitude and phase (the phases of the original and the reflected wave are the same since no phase shift occurs for reflections at open ends),

$$u_2(x,t) = u_0 \cdot \cos\left\{k(x + vt)\right\}. \tag{4.11}$$

The superposition of these waves becomes

$$\begin{aligned} u(x,t) &= u_1(x,t) + u_2(x,t) \\ &= u_0 \cdot [\cos\{k(x-vt)\} \; + \; \cos\{k(x+vt)\}] \\ &= 2u_0 \cdot \cos(kx) \cdot \cos(kvt) \end{aligned} \tag{4.12}$$

This equation describes a standing wave: The spacial and time-wise components of the oscillation are decoupled from each other and one has a stationary wave picture as it is depicted in Fig. 4.3. Every cross sectional area performs an oscillation whose amplitude depends on the position but is constant in time. As a matter of fact, so-called nodes of oscillation emerge at which the oscillation amplitude vanishes, meaning the cross sectional areas at those places are constantly at rest. Between two nodes of oscillation is a vibration antinode where the amplitude becomes maximal. The distance in between two antinodes or two nodes is exactly half of the wavelength of the wave.

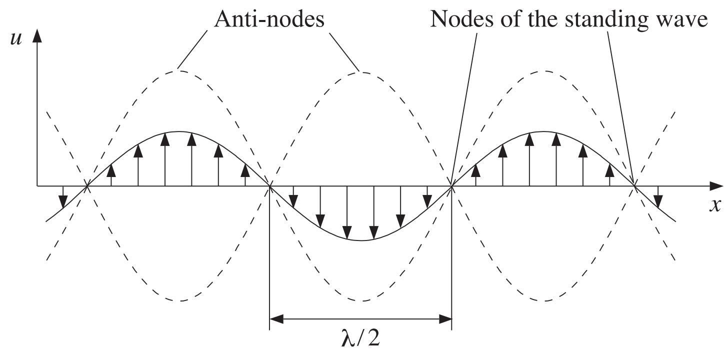

Figure 4.3: Standing wave. The solid line represents the longitudinal displacement at some time t, whereby the dashed line stands for the contour of the oscillation (maximum displacement).

Boundary conditions yield only some very particular wavelength to be allowed. In the middle of the rod has to be an oscillation node since the rod is chucked at this place. Furthermore, antinodes have to lie at both endings of the rod. The wavelength of fundamental oscillation (largest wavelength possible) becomes $\lambda_0 = 2 \cdot l$ and for the n-th harmonic one has

$$\lambda_i = \frac{2}{2n+1} \cdot l \qquad \text{where} \quad n = 0, 1, 2, \ldots \tag{4.13}$$

The locations of the oscillation nodes and antinodes for the fundamental oscillation and the first harmonic are depicted in Fig. 4.4.

So the wavelength of the oscillation can be determined out of the length of the rod, given one has knowledge about which harmonic fundamental oscillation has been stimulated.

The frequency of the standing wave can be determined by converting the sound with a microphone into an electrical signal. The frequency of this signal is then being matched with known frequencies of an alternating voltage. The procedure is being described in the experimental part.

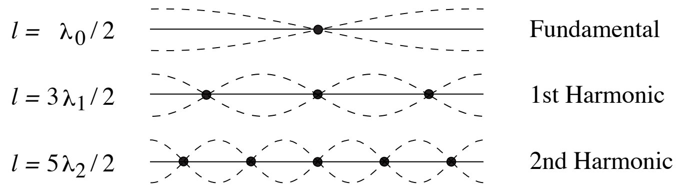

Figure 4.4: Contour of the fundamental and the first two harmonic oscillations. Caution: In this experiment we look at longitudinal waves, the dashed lines illustrate the displacement of the cross sectional areas along the rod!

4.3 Experimental part

In this experiment, you shall determine the modulus of elasticity of aluminum, steel and brass by inducing a standing wave in a center-wise chucked rod and measuring its frequency. The wave lengths of the different vibrational modes are given by Equ. 4.13 and depend on the rods length.

Having the frequency and wavelength, one shall calculate the speed of sound inside the material according to Equ. 4.9 and the modulus of elasticity according to Equ. 4.8. The densities of the said materials are given in table 4.1.

Table 4.1: Density of the used materials.

Material	Density ρ [g/cm^3]
Aluminum	2.71 ± 0.02
Steel	7.86 ± 0.06
Brass	8.43 ± 0.06

a) Experimental set-up and method of measurement

The experimental set-up of the experimental arrangement is depicted in Fig. 4.5. The sound of the oscillating rod is being converted into an electric signal of same frequency and amplified using a microphone. The frequency ν of the amplified signal is then being compared to the adjustable frequency ν' of a generated harmonic alternating voltage which is being created with the help of a frequency generator.

For the comparison, both signal are being connected to the x- and y-inlets of a cathode-ray oscilloscope, such that the horizontal and vertical amplitude of the light point is always determined by one of the two AC voltages (the working principle of a cathode-ray oscilloscope is being discussed in the appendix). The path of the signal describes a Lissajous-figure, whose form is being determined by the applied AC voltages. For $\nu' = \nu$, the AC voltages only differ in phase ϕ. For $\phi = 0$, one has

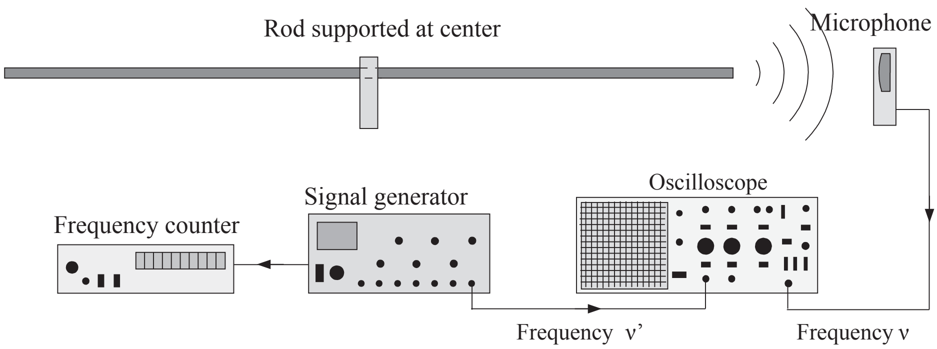

Figure 4.5: Experimental set-up for the determination of the frequency.

$$x(t) = A \cdot \sin(2\pi\nu \cdot t) \quad (4.14)$$
$$y(t) = B \cdot \sin(2\pi\nu \cdot t) \quad (4.15)$$

for the path of the light point and thus

$$\frac{x(t)}{y(t)} = \frac{A}{B} \quad \text{oder} \quad y(t) = \frac{B}{A} \cdot x(t) \quad (4.16)$$

Therefore, the light point describes a straight line. If the relative phase is $\phi = \pi/2$, one has

$$x(t) = A \cdot \sin(2\pi\nu \cdot t) \quad (4.17)$$
$$y(t) = B \cdot \sin(2\pi\nu \cdot t + \pi/2) = B \cdot \cos(2\pi\nu \cdot t) \quad (4.18)$$

and therefore

$$\frac{x^2(t)}{A} + \frac{y^2(t)}{B} = \sin^2(2\pi\nu \cdot t) + \cos^2(2\pi\nu \cdot t) = 1 \quad (4.19)$$

So the light point describes an ellipse. Both cases are illustrated in Fig. 4.6

If the path of the light point describes a straight line or an ellipse, this means, that the adjusted frequency ν' from the frequency generator equals to frequency ν of the acoustic source. Now, one can read off the frequency from the attached meter at the frequency generator.

b) Experimental part

Subsequently, one shall determine the modules of elasticity of brass, aluminum and steel by the previously described method.

- Let the teaching assistant introduce you to the use of the cathode-ray oscilloscope.
- Measure the length of the rods and do an estimation on the accuracy of your measurement.
- Chuck the first rod. Mind, that the rod has to be clamped exactly in the middle.

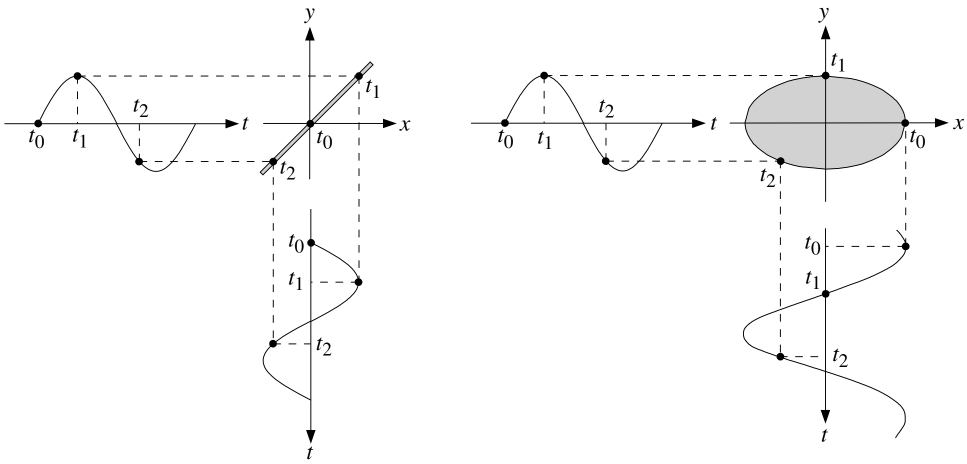

Figure 4.6: Path of the light point for $\nu' = \nu$ and $\phi = 0$ (left), $\phi = \pi/2$ (right)

- Excite a standing wave in the rod and determine in which harmonic the rod is swinging. To do so, count the amount of oscillation nodes by slightly putting your finger on the rod in different places. If you touch the rod at one of its antinodes, the oscillation will abruptly die away. Touching a node, the oscillation is barely affected. After each try, you have to re-excite the wave by rubbing the rod. Determine the wavelength of the standing wave after Equ. 4.13. Estimate the uncertainty.

- Determine the frequency of the oscillation according to the previously discussed method. Perform the measurement five times, while you do a new alignment of the frequency generator for each measurement. Determine the mean and the uncertainty of your measurement.

- Calculate the speed of sound according to Equ. 4.9 and the modulus of elasticity after Equ. 4.8. For it, use the given values for the densities of the materials from table 4.1. Estimate the possible systematic errors and perform an error calculation.

- Repeat the measurements for the second and third rod.

4.4 Appendix

The core of a cathode-ray oscilloscope is a cathode tube, as they are applied in practice in tube televisions.

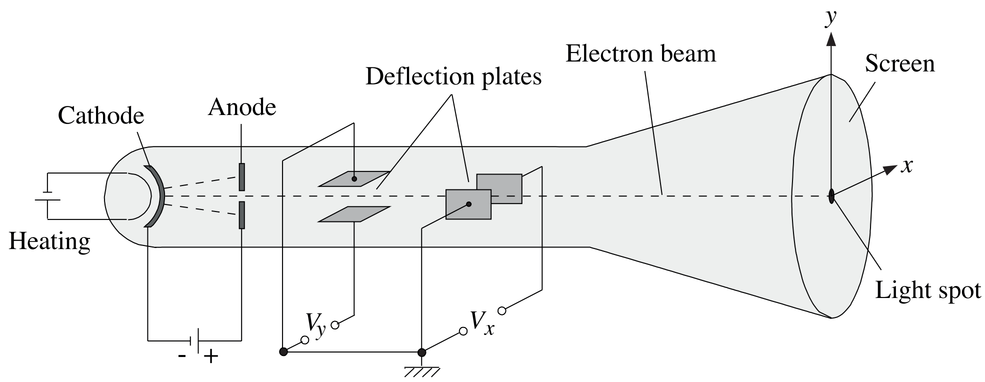

Figure 4.7: Working principle of a cathode tube.

The working principle of a cathode tube is illustrated in Fig. 4.7. Inside of an evacuated glass bulb, an electron ray (cathode-ray) is being generated by accelerating electrons, which escape a hot cathode, towards an anode with a small aperture. The ray, after passing the aperture, successively traverses two orthogonal plate capacitors, where it can be deflected horizontally or vertically. Finally, it hits a fluorescent screen, creating a light point due to fluorescence. The rays horizontal and vertical deflection on the screen is proportional to the applied voltage on the particular plate capacitor($x \propto V_x$ and $y \propto V_y$). If V_x and V_y are not constant in time, the light point describes path, determined by the temporal progression of the two voltages.

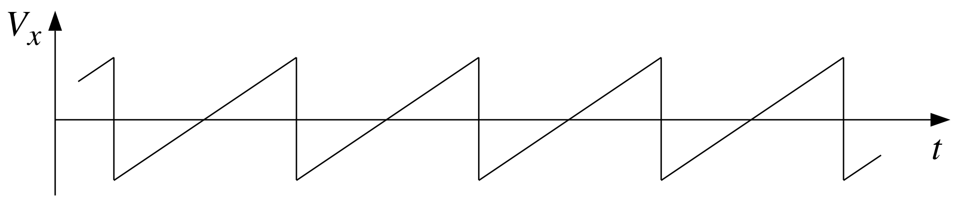

Figure 4.8: Sawtooth voltage.

In particular, if $V_x(t)$ describes a sawtooth voltage, as shown in Fig. 4.8, the light point moves with horizontal velocity from left to right over the whole screen, quickly jumps back to the left side, and so forth. Hence, the path of the light point becomes a direct graphical depiction of the voltage $V_y(t)$ as a function of time.