Advanced Field Theory: Exercise Sheet 8

Due: Thursday, 17.4.2014

Exercise 12 Instead of explicitly integrating out heavy degrees of freedom on the path integral level, an alternative approach to derive the Lagrangian of an effective field theory is to write down a minimal set of operators for each mass dimension and then performing a matching computation – that is to compute a minimal set of quantities in the full theory and in the ffective theory – to determine the coefficients of these operators in the Lagrangian. We will demonstrate how this works for the Euler-Heisenberg effective theory, i.e. the theory obtained after integrating out the electron in QED.

a) Use the charge invariance of QED to show that any operator with an odd number of $F^{\mu\nu}$ is forbidden in the effective theory (this is also known as Furry's theorem.)

b) Write down all gauge, Lorentz and C invariant operators up to mass dimension 8 and show that they can be reduced to the following set:

$$F_{\mu\nu}F^{\mu\nu} \tag{1}$$

$$F_{\mu\nu}\partial^2 F^{\mu\nu} \tag{2}$$

$$F_{\mu\nu}\partial^4 F^{\mu\nu}, \quad \left(F_{\mu\nu}F^{\mu\nu}\right)^2, \quad \left(\varepsilon_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda}\right)^2. \tag{3}$$

Hint: The following identities might be useful:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \tag{4}$$

$$F_{\mu\nu}F^{\nu\lambda}F_{\lambda\rho}F^{\rho\mu} = \frac{1}{2}\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + \frac{1}{16}\left(\varepsilon_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda}\right)^2. \tag{5}$$

c) We found that up to mass dimension 8, the effective Lagrangian has the form

$$\mathcal{L}_{EH} = -\frac{c_0}{4}F_{\mu\nu}F^{\mu\nu} + \frac{c_1}{m_e^2}F_{\mu\nu}\partial^2 F^{\mu\nu} + \frac{1}{m_e^4}\left[c_2^1\left(F_{\mu\nu}F^{\mu\nu}\right)^2 + c_2^2\left(\varepsilon_{\mu\nu\rho\lambda}F^{\mu\nu}F^{\rho\lambda}\right)^2 + c_2^3 F_{\mu\nu}\partial^4 F^{\mu\nu}\right] + \dots \tag{6}$$

To fix c_0 and c_1, compute the one-loop photon vacuum polarization diagram in QED in the $\overline{MS}$-scheme and expand the result in q^2/m_e^2, where q is the photon momentum. You should find

$$i\left(q_\mu q\nu - q^2 g_{\mu\nu}\right)\Pi^0(q^2), \tag{7}$$

$$\Pi^0(q^2) = -\frac{2\alpha}{\pi}\left(\frac{1}{6}\ln\frac{m_e^2}{\mu^2} + \int_0^1 dx\, x(1-x)\ln\left(1 - x(1-x)\frac{q^2}{m_e^2}\right)\right) \tag{8}$$

$$= -\frac{\alpha}{3\pi}\ln\frac{m_e^2}{\mu^2} + \frac{\alpha}{15\pi}\frac{q^2}{m_e^2} + \mathcal{O}\left(\frac{q^4}{m_e^4}\right), \tag{9}$$

where μ is the renormalization scale.

d) Why is it advisable to perform the matching at a scale close to the electron mass?

e) Show that we obtain

$$c_0(\mu = m_e) = 1 \tag{10}$$

$$c_1(\mu = m_e) = \frac{\alpha}{60\pi}. \tag{11}$$