Advanced Field Theory: Exercise Sheet 7

Due: Thursday, 10.4.2014

Exercise 10 The linear sigma model consists of two Dirac fermions that we arrange in a vector $\psi = \begin{pmatrix} u \\ d \end{pmatrix}$ and four real scalars σ and $\vec{\pi} = (\pi^1, \pi^2, \pi^3)$ descibed by the Lagrangian density

$$\mathcal{L}_\sigma^{\text{lin}} = \frac{1}{2}(\partial_\mu \sigma)(\partial^\mu \sigma) + \frac{1}{2}(\partial_\mu \vec{\pi}) \cdot (\partial^\mu \vec{\pi}) - \frac{\lambda}{4}(\sigma^2 + \vec{\pi} \cdot \vec{\pi} - v^2)^2 + \bar{\psi} i \partial\!\!\!/ \psi + g\bar{\psi}(\sigma + i\vec{\sigma} \cdot \vec{\pi}\gamma_5)\psi. \tag{1}$$

a) Show that (1) can be recast as

$$\mathcal{L}_\sigma^{\text{lin}} = \frac{1}{4}\text{tr}\,(\partial_\mu \Sigma)^\dagger (\partial^\mu \Sigma) - \frac{\lambda}{4}\left[\frac{1}{2}\text{tr}\,\Sigma^\dagger \Sigma - v^2\right]^2 + \bar{\psi}_L i \partial\!\!\!/ \psi_L + \bar{\psi}_R i \partial\!\!\!/ \psi_R + g\bar{\psi}_L \Sigma \psi_R + g\bar{\psi}_R \Sigma^\dagger \psi_L, \tag{2}$$

where $\Sigma \equiv \sigma \mathbb{1} + i\vec{\tau} \cdot \vec{\pi}$ and $\psi_{R,L}$ are the right-handed and left-handed parts.

The fields $\psi_{R,L}$ and Σ transform accrording to

$$\psi_{R,L} \to U_{L,R}\psi_{R,L}, \quad \Sigma \to U_L \Sigma U_R^\dagger, \qquad U_{R,L} \equiv \exp\left(-i\frac{1}{2}\vec{\alpha}_{R,L} \cdot \vec{\tau}\right) \in \text{SU}(2). \tag{3}$$

b) Using the properties of the Pauli matrices, recover σ and $\vec{\pi}$ from Σ and show that they transform as

$$\sigma \to \sigma + \frac{1}{2}\left(\vec{\alpha}_L - \vec{\alpha}_R\right) \cdot \vec{\pi}, \tag{4}$$

$$\pi^k \to \pi^k - \frac{1}{2}\left(\alpha_L^k - \alpha_R^k\right)\sigma - \frac{1}{2}\epsilon^{klm}\pi^l \left(\alpha_L^m + \alpha_R^m\right). \tag{5}$$

Exercise 11

a) Using the properties of the Pauli matrices, prove that

$$U \equiv \exp\left(i\frac{\vec{\tau} \cdot \vec{\xi}}{v}\right) = \cos\left(\frac{\xi}{v}\right) + i\frac{\vec{\tau} \cdot \vec{\xi}}{\xi}\sin\left(\frac{\xi}{v}\right), \quad \xi = |\vec{\xi}|. \tag{6}$$

The Lagrangian of the non-linear sigma model contains the term

$$\mathcal{L}_\sigma^{\text{non-lin}} \supset \Delta\mathcal{L} = \frac{1}{4}(v + S)^2 \text{tr}\,(\partial_\mu U)^\dagger (\partial^\mu U). \tag{7}$$

b) Using (6), express $\Delta\mathcal{L}$ with S and $\vec{\xi}$ retaining only the terms up to $\mathcal{O}(1/v)$.

Hint: You might find the following relation useful: $\partial_\mu \xi = (\vec{\xi} \cdot \partial_\mu \vec{\xi})/\xi$.

c) What kind of vertices do these terms represent?

d) Show that up to $\mathcal{O}(1/v)$ the tree-level amplitudes for Goldstone boson scattering $\mathcal{M}_{\pi^+\pi^0 \to \pi^+\pi^0}$ agree in the two representations (1) and (7).

e) Show that the (nonlinear) field redefinition

$$\Sigma = (S + v)\,U \tag{8}$$

satisfies the condition of the Haag theorem, that is, show that the Jacobian of the coordinate transformation is non-degenerate at the origin.