Advanced Field Theory: Exercise Sheet 5

Due: Thursday, 27.3.2014

Exercise 7 We want to compute the width of the $\pi^+ \to \ell^+ \nu_\ell$ decay. The relevant Lagrangian is given by

$$\mathcal{L}_{int} = \frac{4G_F}{\sqrt{2}} \left(\bar{\ell}_L \gamma^\mu \nu_L\right) \left(\bar{u}_L \gamma_\mu d_L\right) + \text{h.c.}, \tag{1}$$

where $\nu_L = P_L \nu_L = 1/2\left(1-\gamma^5\right)\nu$.

a) Give a qualitative argument – based on angular momentum conservation and the $V-A$ structure of weak interactions – why the semi-leptonic pion decay is forbidden for vanishing lepton masses $m_\ell = 0$.

b) Defining

$$J^{\mu a} = \bar{Q}\gamma^\mu \tau^a Q, \qquad J^{\mu 5 a} = \bar{Q}\gamma^\mu \gamma^5 \tau^a Q, \tag{2}$$

where $Q = \begin{pmatrix} u \\ d \end{pmatrix}$ is the quark doublet and $\tau^a = \sigma^a/2$ are the generators of $SU(2)$, show that

$$\bar{u}_L \gamma^\mu d_L = \frac{1}{2}\left(J^{\mu 1} + iJ^{\mu 2} - J^{\mu 51} - iJ^{\mu 52}\right). \tag{3}$$

c) The matrix element of $J^{\mu 5a}$ between the vacuum and an on-shell pion can be written as

$$\langle 0|J^{\mu 5a}|\pi^b(p)\rangle = -ip^\mu f_\pi \delta^{ab} e^{-ip\cdot x} \tag{4}$$

where f_π is a constant with the dimension of a mass.

d) Using this identification together with the result of part b, show that the amplitude for the decay $\pi^+ \to \ell^+\nu$ ($|\pi^+\rangle = \frac{1}{\sqrt{2}}(|\pi^1\rangle + i\,|\pi^2\rangle)$) is given by

$$i\mathcal{M} = G_F f_\pi \bar{u}(q) \not{p}(1-\gamma^5) v(k) \tag{5}$$

where p, k and q are the momenta of the π^+, ℓ^+ and ν respectively.

Hint: Write down $\mathcal{L}_{int}$ in terms of $J^{\mu a}$ and $J^{\mu 5a}$ and compute $\langle out|L_{int}|in\rangle$.

e) Show that the sum over final-state spins gives

$$\sum_{spins} |\mathcal{M}|^2 = 4G_F^2 f_\pi^2 m_\pi^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2}\right). \tag{6}$$

f) The decay width of a particle of mass m through the channel corresponding to $\mathcal{M}$ is given by

$$\Gamma = \frac{1}{2m}\int d\Phi_{1\to 2} |\mathcal{M}|^2, \tag{7}$$

where the phase-space measure is

$$d\Phi_{1\to 2} = \frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4}(2\pi)\delta(p_1^2 - m_1^2)(2\pi)\delta(p_2^2 - m_2^2)(2\pi)^4\delta^{(4)}(p - p_1 - p_2). \tag{8}$$

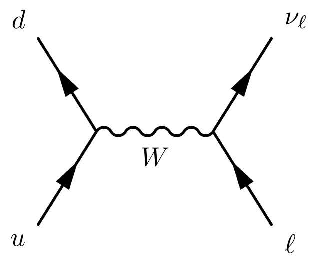

Figure 1: Feynman diagram for the semi-leptonic pion decay.

Show that

$$\Gamma(\pi^+ \to \ell^+ \nu_\ell) = \frac{1}{4\pi} G_F^2 f_\pi^2 m_\pi m_\ell^2 \left(1 - \frac{m_\ell^2}{m_\pi^2}\right). \tag{9}$$

In particular, the decay width vanishes for a vanishing lepton mass and we have the f_π-independent prediction

$$\frac{\Gamma(\pi^+ \to e^+ \nu)}{\Gamma(\pi^+ \to \mu^+ \nu)} = \left(\frac{m_e}{m_\mu}\right)^2 \frac{\left(1 - m_e^2/m_\pi^2\right)^2}{\left(1 - m_\mu^2/m_\pi^2\right)^2} \approx 10^{-4}. \tag{10}$$

g) From the measured pion lifetime, the Fermi constant G_F, and the pion and muon masses determine the value of f_π.

Hint: You might want to take a look at http://pdg.lbl.gov/.