Advanced Field Theory: Exercise Sheet 3

Due: Thursday, 13. 3. 2014

Exercise 5 We will study the non-conservation of the axial current j_5^μ in two-dimensional QED, described by the Lagrangian

$$\mathcal{L} = i\bar{\Psi}\not{D}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

where $\mu, \nu = 0, 1$ and $D_\mu = \partial_\mu + ieA_\mu$. The Dirac algebra is the same as in four dimensions:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \tag{2}$$

where the γ^μ are now 2×2 matrices.

We will consider the classically conserved axial current $j_5^\mu = \bar{\Psi}\gamma^\mu\gamma_5\Psi$.

a) To see where the non-conservation at the quantum level comes from, we have to take into account that the product of two local fields at the same spacetime point might be singular. To regularize this singularity, we can consider the product of $\bar{\Psi}$ and Ψ at two different points separated by a small vector ε^μ and take the limit $\varepsilon \to 0$. The naive choice would be to define

$$j_5^\mu = \operatorname{symm}\lim_{\varepsilon\to 0} \left\{ \bar{\Psi}(x+\varepsilon/2)\gamma^\mu\gamma_5\Psi(x-\varepsilon/2) \right\}, \tag{3}$$

where symm lim specifies that the limit should be taken symmetrically, i.e.

$$\operatorname{symm}\lim_{\varepsilon\to 0} \left\{ \frac{\varepsilon^\mu}{\varepsilon^2} \right\} = 0, \quad \operatorname{symm}\lim_{\varepsilon\to 0} \left\{ \frac{\varepsilon^\mu\varepsilon^\nu}{\varepsilon^2} \right\} = \frac{1}{d}g^{\mu\nu}, \tag{4}$$

with $d = 2$ in this case, which is necessary to preserve Lorentz invariance

Show that j_5^μ as defined in Eq. (3) is not gauge invariant.

b) Gauge invariance can be restored by adding a so-called *Wilson line*:

$$j_5^\mu = \operatorname{symm}\lim_{\varepsilon\to 0} \left\{ \bar{\Psi}(x+\varepsilon/2)\gamma^\mu\gamma_5 \exp\left[-ie\int_{x-\varepsilon/2}^{x+\varepsilon/2} dz\cdot A(z)\right]\Psi(x-\varepsilon/2) \right\}. \tag{5}$$

Show explicitly that j_5^μ as defined in Eq. (5) is in fact gauge invariant.

c) Show that after using the equations of motions

$$\not{\partial}\Psi = -ie\not{A}\Psi, \quad \partial_\mu\bar{\Psi}\gamma^\mu = ie\bar{\Psi}\not{A} \tag{6}$$

and to first order in ε, the divergence of the axial current reads

$$\partial_\mu j_5^\mu = \operatorname{symm}\lim_{\varepsilon\to 0} \left\{ \bar{\Psi}(x+\varepsilon/2)\left[-ie\gamma^\mu\varepsilon^\nu\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\right]\gamma_5\Psi(x-\varepsilon/2) \right\}. \tag{7}$$

d) The singular part of the product of two fields is given by the Feynman propagator (the Wick contraction), i.e. in this case by

$$\Psi(y)\bar{\Psi}(z) = -\frac{i}{2\pi}\frac{\gamma^\lambda\,(y-z)_\lambda}{(y-z)^2} + \ldots, \tag{8}$$

that is

$$\bar{\Psi}(x+\varepsilon/2)\Gamma\Psi(x-\varepsilon/2) = -\frac{i}{2\pi}\mathrm{tr}\left[\frac{\gamma^\lambda \varepsilon_\lambda}{\varepsilon^2}\Gamma\right] + \ldots, \tag{9}$$

where Γ is any combination of γ matrices and the dots represent additional non-singular terms. Use Eq. (9) and

$$\mathrm{tr}\,[\gamma^\mu \gamma^\nu \gamma_5] = 2\varepsilon^{\mu\nu} \tag{10}$$

to show that

$$\begin{aligned} \partial_\mu j_5^\mu &= \frac{e}{2\pi}\,\mathrm{symm}\lim_{\varepsilon\to 0}\left\{2\frac{\varepsilon_\mu \varepsilon^\nu}{\varepsilon^2}\right\}\varepsilon^{\mu\lambda}F_{\nu\lambda} \\ &= \frac{e}{2\pi}\varepsilon^{\mu\nu}F_{\mu\nu}. \end{aligned} \tag{11}$$

Note that we could have sacrificed the gauge invariance by reversing the sign of the Wilson line in Eq. (5) and would have ended up with a conserved axial current, showing again the tension between vector current conservation (gauge invariance) and axial vector conservation.