Advanced Field Theory: Exercise Sheet 11

Due: Thursday, 22.5.2014

Exercise 15 We will derive the Adler-Bell-Jackiw anomaly from the non-invariance of the functional measure $\mathcal{D}\psi\mathcal{D}\bar{\psi}$ in the fermionic functional integral

$$Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \exp\left[i \int d^4x\, \bar{\psi}\left(i\not{D}\right)\psi\right] \tag{1}$$

under axial transformations,

$$\psi(x) \to \psi'(x) = \left(1 + i\alpha(x)\gamma^5\right)\psi(x) \tag{2}$$

$$\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x)\left(1 + i\alpha(x)\gamma^5\right). \tag{3}$$

The Lagrangian is manifestly invariant under this transformation. To analyze the functional measure, we expand the fermionic fields in terms of right and left eigenstates ϕ_m and $\hat{\phi}_m$ of $\not{D}$:

$$\left(i\not{D}\right)\phi_m = \lambda_m\phi_m, \qquad \hat{\phi}_m\left(i\not{D}\right) \equiv -iD_\mu\hat{\phi}_m\gamma^\mu = \lambda_m\hat{\phi}_m. \tag{4}$$

Then for ψ and $\bar{\psi}$

$$\psi(x) = \sum_m a_m\phi_m(x), \qquad \bar{\psi}(x) = \sum_m \hat{a}_m\hat{\psi}_m(x), \tag{5}$$

where a_m and $\hat{a}_m$ are anticommuting coefficients. The functional measure can be defined as

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = \prod_m da_m d\hat{a}_m. \tag{6}$$

a) Show that

$$a'_m = \sum_n \int d^4x\, \phi^\dagger_m(x)\left(1 + i\alpha(x)\gamma^5\right)\phi_n(x)a_n \equiv \sum_n \left(\delta_{mn} + C_{mn}\right)a_n, \tag{7}$$

where a'_m are the expansion coefficients of ψ'.

The functional measure transforms as (remember the rules for a change of integration variables when integrating over fermionic fields)

$$\mathcal{D}\psi'\mathcal{D}\bar{\psi}' = \mathcal{J}^{-2}\mathcal{D}\psi\mathcal{D}\bar{\psi}, \tag{8}$$

where $\mathcal{J}$ is the Jacobian of $(1 + C)$,

$$\mathcal{J} = \det(1 + C). \tag{9}$$

b) Show that in leading order in α, $\mathcal{J}$ can be expressed as

$$\log\mathcal{J} = \lim_{M\to\infty} i \int d^4x\, \alpha(x) \sum_n \phi^\dagger_n(x)\gamma^5\phi_n(x)e^{\lambda_n^2/M^2} \tag{10}$$

$$= \lim_{M\to\infty} i \int d^4x\, \alpha(x)\langle x|\mathrm{tr}\left[\gamma^5 e^{(i\not{D})^2/M^2}\right]|x\rangle. \tag{11}$$

c) Show that

$$\lim_{M\to\infty} \langle x|\mathrm{tr}\left[\gamma^5 e^{(i\not{D})^2/M^2}\right]|x\rangle = \lim_{M\to\infty} \mathrm{tr}\left[\gamma^5 \frac{1}{2}\left(\frac{e}{2M^2}\sigma^{\mu\nu}F_{\mu\nu}\right)^2\right]\langle x|e^{-\partial^2/M^2}|x\rangle, \tag{12}$$

where $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$.

Hint: Use $(i\not{D})^2 = -D^2 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}$.

d) Show that

$$\langle x|e^{-\partial^2/M^2}|x\rangle = i\frac{M^4}{16\pi^2} \tag{13}$$

and thus

$$\mathcal{J} = \exp\left[-i\int d^4x\,\alpha(x)\left(\frac{e^2}{32\pi^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}(x)F_{\lambda\sigma}(x)\right)\right], \tag{14}$$

i.e. for the functional integral

$$Z = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left[i\int d^4x\,\bar{\psi}\left(i\not{D}\right)\psi + \alpha(x)\left\{\partial_\mu j^{5\mu} + \frac{e^2}{16\pi^2}\epsilon^{\mu\nu\lambda\sigma}F_{\mu\nu}(x)F_{\lambda\sigma}(x)\right\}\right]. \tag{15}$$

Varying the exponent with respect to α will give the known result for the Adler-Bell-Jackiw anomaly.