Advanced Field Theory: Exercise Sheet 10

Due: Thursday, 15.5.2014

Exercise 14 Consider a SU(N) gauge theory, with the covariant derivative defined as

$$D_\mu \equiv \partial_\mu + igA_\mu, \quad A_\mu \equiv A_\mu^a T^a, \quad F_{\mu\nu}^a \equiv \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f^{abc} A_\mu^b A_\nu^c, \tag{1}$$

where the T^a are the generators of SU(N), satisfying $[T^a, T^b] = i f^{abc} T^c$.

a) Show that $F_{\mu\nu}^a \tilde{F}^{\mu\nu,a} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^a F_{\rho\sigma}^a$ can be written as the total derivative of the *Chern-Simons current*,

$$K^\mu = \varepsilon^{\mu\nu\rho\sigma}\left(A_\nu^a F_{\rho\sigma}^a + \frac{g}{3} f_{abc} A_\nu^a A_\rho^b A_\sigma^c\right) \tag{2}$$

The *topological charge* associated to K^μ is defined as

$$n \equiv \frac{g^2}{32\pi^2}\int d^3x\, K_0. \tag{3}$$

Consider an adiabatic transformation

$$A_\mu(t = -\infty) = 0 \tag{4}$$

$$A_\mu(t = +\infty) = \frac{i}{g}\left(\partial_\mu \Lambda\right)\Lambda^{-1} \qquad \Lambda = \frac{\vec{x}^2 - d^2}{\vec{x}^2 + d^2}\mathbb{1} + i\frac{2d}{\vec{x}^2 + d^2} x_k \tau_k, \tag{5}$$

where the τ_i denote as usual the generators of SU(2): $\tau_i\tau_j = \delta_{ij}\mathbb{1} + i\varepsilon_{ijk}\tau_k$.

b) Check that $F_{\mu\nu} = 0$, so that the topological charge reads

$$n = \frac{g^3}{96\pi^2}\int d^3x\; \varepsilon^{ijk} f^{abc} A_i^a A_j^b A_k^c = -i\frac{g^3}{24\pi^2}\int d^3x\; \varepsilon^{ijk}\mathrm{tr}\, A_i A_j A_k. \tag{6}$$

Hint: Express f^{abc} from a trace to obtain the last equality.

c) Show that

$$A_i(t = +\infty) = \frac{-2\,d}{g(\vec{x}^2 + d^2)^2}\left[(\vec{x}^2 - d^2)\tau_i - 2(x_j\tau_j)x_i + 2d\,\varepsilon_{ijk}x_j\tau_k\right]. \tag{7}$$

d) Compute the topological charge n of this adiabatic transformation.

Hint: Recall the contraction identities for Levi-Civita tensors.