Advanced Field Theory: Exercise Sheet 1

Due: Thursday, 27. 2. 2014

Exercise 1 We consider source-free electrodynamics, i.e. the theory described by the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \tag{1}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

is the electromagnetic field strength.

a) For a general coordinate and field transformation such that

$$A^\mu \to A^\mu + \alpha \delta A^\mu \tag{3}$$

$$\mathcal{L} \to \mathcal{L} + \alpha \partial_\mu \mathcal{J}^\mu, \tag{4}$$

where α is an infinitesimal parameter, the conserved Noether current reads

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu A^\nu\right)} \delta A^\nu - \mathcal{J}^\mu. \tag{5}$$

Show that for translations, that is for transformations of the form

$$A^\mu \to A^\mu + \alpha^\nu \partial_\nu A^\mu \tag{6}$$

$$\mathcal{L} \to \mathcal{L} + \alpha^\nu \partial_\nu \mathcal{L} = \mathcal{L} + \alpha^\nu \partial_\mu \left(\delta^\mu{}_\nu \mathcal{L}\right), \tag{7}$$

one obtains four conserved currents $(j^\nu)^\mu \equiv T^{\mu\nu}_{can}$ given by

$$T^{\mu\nu}_{can} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu A^\lambda)} \partial^\nu A^\lambda - \eta^{\mu\nu} \mathcal{L}. \tag{8}$$

The tensor $T^{\mu\nu}_{can}$ is called the *energy-momentum tensor*.

b) Show explicitly that $T^{\mu\nu}_{can}$ is neither symmetric nor gauge invariant.

c) The general form of the *Belinfante tensor* $T^{\mu\nu}$ reads

$$T^{\mu\nu} = T^{\mu\nu}_{can} + \frac{1}{2} \partial_\kappa \left[\frac{\partial \mathcal{L}}{\partial \left(\partial_\kappa \Psi^l\right)} \left(I^{\mu\nu}\right)^l{}_m \Psi^m - \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \Psi^l\right)} \left(I^{\kappa\nu}\right)^l{}_m \Psi^m - \frac{\partial \mathcal{L}}{\partial \left(\partial_\nu \Psi^l\right)} \left(I^{\kappa\mu}\right)^l{}_m \Psi^m \right], \tag{9}$$

where the matrices $I^{\mu\nu}$ encode the transformation of the fields ψ^l under infinitesimal Lorentz transformations:

$$x^\mu \to x^\mu + \omega^\mu{}_\nu x^\nu \tag{10}$$

$$\Psi^l \to \Psi^l + \frac{1}{2} \omega^{\mu\nu} \left(I_{\mu\nu}\right)^l{}_m \Psi^m. \tag{11}$$

For spin-1 (vector) fields A^μ, the matrices $I^{\mu\nu}$ have the form

$$\left(I_{\mu\nu}\right)^\kappa{}_\sigma = \delta_\mu{}^\kappa \eta_{\nu\sigma} - \delta_\nu{}^\kappa \eta_{\mu\sigma}. \tag{12}$$

Show that $T^{\mu\nu}$ can be written in a manifestly symmetric and gauge invariant way as

$$T^{\mu\nu} = \eta_{\kappa\sigma} F^{\kappa\mu} F^{\nu\sigma} - \eta^{\mu\nu} \mathcal{L}. \tag{13}$$

As it is also conserved, it can be used instead of $T^{\mu\nu}_{can}$ as ‘the’ energy-momentum tensor.

d) Use $E^i = -F^{0i}$ and $\varepsilon^{ijk} B^k = F^{ji}$ to express $T^{\mu\nu}$ in terms of the electric and magnetic fields and show that it takes its classically expected form

$$\mathcal{E} \equiv T^{00}, \qquad S^k \equiv T^{0k}, \quad \text{where} \tag{14}$$

$$\mathcal{E} = \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right), \qquad \vec{S} = \vec{E} \times \vec{B}. \tag{15}$$