(1)

ANOMALIES

Let us consider an action possessing a certain invariance at the classical level. If this invariance cannot be preserved at the quantum level (that is, taking into account quantum corrections) such a phenomenon is called "anomaly".

We have already seen an example of anomaly: the breaking of scale invariance. A theory which does not contain an explicit mass term in the lagrangian leads to a conserved classical dilatation current. When quantum corrections are switched on, the necessity to regularize and renormalize the theory necessarily introduces a mass scale and the dilatation current is not conserved any more.

There are two types of anomalies. Following Shifman, we call them internal and external. In the first case the symmetry we deal with is a gauge symmetry: such anomaly leads to a breaking of gauge invariance and this is a disaster for the theory, since the theory cannot be consistently quantized.

External anomalies also result in current non-conservation. In this case, however, the anomalous current is not connected to a gauge symmetry and this breaking of the symmetry usually has interesting (important) consequences.

In the following we will discuss the CHIRAL ANOMALY

Chiral Symmetry*

Let us consider the lagrangian for a massless fermion $\mathcal{L} = i\bar{\psi}\not{D}\psi$ coupled with a $U(1)$ field A_μ $\quad D_\mu = \partial_\mu + ieA_\mu$

- The vector transformation $\psi \to e^{i\vartheta}\psi$ is a symmetry

$$\psi \to e^{i\vartheta} \qquad \bar{\psi} \to e^{-i\vartheta} \qquad\qquad \bar{\psi}\not{D}\psi \to \bar{\psi}e^{-i\vartheta}\not{D}e^{i\vartheta}\psi = \bar{\psi}\not{D}\psi$$

The associated Noether current is

$$\delta J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu\psi)}\,\delta\psi = i\bar{\psi}\gamma^\mu i\vartheta\psi \sim \bar{\psi}\gamma^\mu\psi \qquad \text{vector current}$$

②

- The axial transformation $\quad (\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \quad \gamma_5^+ = \gamma_5 \quad \{\gamma_5, \gamma_\mu\} = 0$

$$\psi \to e^{i\vartheta\gamma_5}\psi$$

is also a symmetry

$$\psi^+ \to \psi^+ e^{-i\vartheta\gamma_5} \quad \Rightarrow \quad \bar\psi \to \psi^+ e^{-i\vartheta\gamma_5}\gamma^0 = \bar\psi e^{i\vartheta\gamma_5}$$

$$\bar\psi \not{D}\psi \to \bar\psi e^{i\gamma_5\vartheta}\not{D} e^{i\gamma_5\vartheta}\psi = \bar\psi \not{D} e^{-i\gamma_5\vartheta} e^{i\gamma_5\vartheta}\psi = \bar\psi\not{D}\psi$$

This symmetry is called <u>CHIRAL SYMMETRY</u> and its conserved current is $\bar\psi\gamma^\mu\gamma_5\psi$

Note that if we add a mass term $m\bar\psi\psi$ this term is still invariant under the vector symmetry but <u>NOT UNDER THE AXIAL</u> symmetry

$$m\bar\psi\psi \to m\bar\psi e^{i\gamma_5\vartheta} e^{i\gamma_5\vartheta}\psi \neq m\bar\psi\psi$$

We can now consider the Green function

$$\langle 0| T J_\mu^5(z) J_\nu(x) J_\lambda(y)|0\rangle$$

where J_μ and J_μ^5 are the vector and axial current, respectively.

The Fourier transform of this Green function is

$$\Gamma^5_{\mu\nu\lambda}(P,K)\,(2\pi)^4\,\delta^4(q-P-K) = \int d^4x\, d^4y\, d^4z\; e^{i(qz-Px-Ky)} \langle 0|T J_\mu^5(z) J_\nu(x) J_\lambda(y)|0\rangle$$

If we take the derivative with respect to z naive current conservation leads to

$$\partial_z^\mu \langle 0|T J_\mu^5(z) J_\nu(x) J_\lambda(y)|0\rangle = \langle 0|T\, \partial_z^\mu J_\mu^5(z) J_\nu(x) J_\lambda(y)|0\rangle = 0$$

(note that, computing the derivative of the time ordered product we have to differentiate also the θ function that will lead to equal time commutators that vanish

$$[J_0(x,t), J_\mu(y,t)] = [J_0^5(x,t), J_\mu(y,t)] = 0$$

The naive conservation of the vector and axial vector currents imply that

$$(P+k)^\mu \, \Gamma^5_{\mu\nu\lambda}(P,k) = P^\nu \, \Gamma^5_{\mu\nu\lambda}(P,k) = k^\lambda \, \Gamma^5_{\mu\nu\lambda}(P,k) = 0$$

NAIVE WARD IDENTITIES

It turns out that it is IMPOSSIBLE to preserve all these three identities

The diagrams contributing to $\Gamma^5_{\mu\nu\lambda}(P,k)$ are

$$\sim \int \frac{d^4 \ell}{\ell^3}$$

↳ superficially linearly divergent in the UV

As a function of P and k $\Gamma^5_{\mu\nu\lambda}(P,k)$ must be a rank three pseudotensor $\Rightarrow$ by using Lorentz invariance we can write

$$\Gamma^5_{\mu\nu\lambda}(P,k) = A_1(P,k)\,\varepsilon_{\mu\nu\lambda\sigma}P^\sigma + A_2\,\varepsilon_{\mu\nu\lambda\sigma}k^\sigma + B_1(P,k)\,P_\lambda\,\varepsilon_{\mu\nu\alpha\beta}P^\alpha k^\beta$$
$$+ B_2(P,k)\,k_\lambda\,\varepsilon_{\mu\nu\alpha\beta}P^\alpha k^\beta + B_3(P,k)\,P_\nu\,\varepsilon_{\mu\lambda\alpha\beta}P^\alpha k^\beta$$
$$+ B_4(P,k)\,k_\nu\,\varepsilon_{\mu\lambda\alpha\beta}P^\alpha k^\beta$$

By dimensional analysis we see that $A_1(P,k)$ and $A_2(P,k)$ are divergent (actually only logarithmically) and that the other form factors are finite.

Let us impose vector current conservation:

$$P^\nu\,\Gamma^5_{\mu\nu\lambda}(P,k) = A_2(P,k)\,\varepsilon_{\mu\nu\lambda\sigma}\,\overset{\nearrow\alpha}{P^\nu}\overset{\nearrow\beta}{k^\sigma} + B_3(P,k)\,P^2\,\varepsilon_{\mu\lambda\alpha\beta}P^\alpha k^\beta$$
$$+ B_4(P,k)\,\varepsilon_{\mu\lambda\alpha\beta}P^\alpha k^\beta\,(P\cdot k)$$
$$= \left(-A_2(P,k) + P^2 B_3(P,k) + (P\cdot k)\,B_4(P,k)\right)\varepsilon_{\mu\lambda\alpha\beta}P^\alpha k^\beta = 0$$

$$k^\lambda\,\Gamma^5_{\mu\nu\lambda}(P,k) = A_1(P,k)\,\varepsilon_{\mu\nu\lambda\sigma}\,\overset{\beta}{\overset{\uparrow}{k^\lambda}}\,\overset{\alpha}{\overset{\uparrow}{P^\sigma}} + B_1(P,k)\,\varepsilon_{\mu\nu\alpha\beta}P^\alpha k^\beta\,(P\cdot k)$$
$$+ B_2(P,k)\,k^2\,\varepsilon_{\mu\nu\alpha\beta}P^\alpha k^\beta$$

$$\Rightarrow \begin{cases} -A_2(p,k) + p^2 B_3(p,k) + (p\cdot k)\, B_4(p,k) = 0 \\ -A_1(p,k) + k^2 B_2(p,k) + (p\cdot k)\, B_1(p,k) = 0 \end{cases}$$

$\Rightarrow$ renormalized A_1 and A_2 are determined in terms of finite B_i

We also note that there is another constraint from Bose symmetry:

$$\Gamma^5_{\mu\nu\lambda}(p,k) = \Gamma^5_{\mu\lambda\nu}(k,p)$$

$$\Rightarrow \quad A_1(p,k)\, \varepsilon_{\mu\nu\lambda\sigma} p^\sigma + A_2(p,k)\, \varepsilon_{\mu\nu\lambda\sigma} k^\sigma = A_1(k,p)\, \varepsilon_{\mu\lambda\nu\sigma} k^\sigma + A_2(k,p)\, \varepsilon_{\mu\lambda\nu\sigma} p^\sigma$$

We conclude that $A_2(k,p) = -A_1(p,k)$ and that only one of the two equations is independent.

→ We should now compute $(p+k)^\mu \Gamma^5_{\mu\nu\lambda}(p,k)$: the anomaly can be computed in many different ways. For the moment we anticipate that

$$\boxed{-(p+k)^\mu\, \Gamma^5_{\mu\nu\lambda}(p,k) = -\frac{i}{2\pi^2}\, \varepsilon_{\mu\lambda\rho\nu}\, k^\mu p^\rho}$$

which implies that the axial current is not conserved (we will prove this result later)

Let us note that the anomaly in the axial current can be eliminated through a finite renormalization of the form factors A_1 and A_2

$$A_1 \to A_1 + a_1$$
$$A_2 \to A_2 + a_2$$

$$-(p+k)^\mu\, \Gamma^5_{\mu\nu\lambda}(p,k) = -\frac{i}{2\pi^2}\, \varepsilon_{\mu\lambda\rho\nu}\, k^\mu p^\rho - a_1\, \varepsilon_{\mu\nu\lambda\sigma} p^\sigma k^\mu - a_2\, \varepsilon_{\mu\nu\lambda\sigma} k^\sigma p^\mu$$

$\Rightarrow$ we can set $(p+k)^\mu \Gamma^5_{\mu\nu\lambda}(p,k) = 0$ by choosing $a_1 - a_2 = -\frac{i}{2\pi^2}$

which by Bose symmetry implies $a_1 = -\frac{i}{4\pi^2}$ $\quad a_2 = \frac{i}{4\pi^2}$

(5) However we can easily check that this finite renormalization moves the anomaly to the vector current!

$$\Gamma^5_{\mu\nu\lambda}(p,k) \to \Gamma^5_{\mu\nu\lambda}(p,k) + a_1 \varepsilon_{\mu\nu\lambda\sigma} p^\sigma + a_2 \varepsilon_{\mu\nu\lambda\sigma} k^\sigma$$

$$p^\nu \Gamma_{\mu\nu\lambda}(p,k) = a_2 \varepsilon_{\mu\nu\lambda\sigma} p^\nu k^\sigma = \frac{i}{4\pi^2} \varepsilon_{\mu\nu\lambda\sigma} p^\nu k^\sigma$$

$$k^\lambda \Gamma_{\mu\nu\lambda}(p,k) = a_1 \varepsilon_{\mu\nu\lambda\sigma} p^\sigma k^\lambda = -\frac{i}{4\pi^2} \varepsilon_{\mu\nu\lambda\sigma} p^\sigma k^\lambda$$

$\Rightarrow$ IT IS IMPOSSIBLE TO SIMULTANEOUSLY SATISFY ALL THE THREE WARD IDENTITIES

The choice has to be made according to the problem under consideration

- EXAMPLE 1

$$\mathcal{L} = \bar{\Psi} i \not{\partial} \Psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad \text{QED with one fermion}$$

The vector current is coupled to the gauge field, whereas the axial current is NOT

$\Rightarrow$ for the Green function $\langle 0 | J^5_\mu(z) J_\nu(x) J_\lambda(y) | 0 \rangle$

we must impose the conditions that satisfy the conservation of the VECTOR current

$\Rightarrow$ AXIAL CURRENT IS NOT CONSERVED $\qquad \partial^\mu J^5_\mu = \frac{g^2}{16\pi^2} \varepsilon^{\mu\nu\rho\delta} F_{\mu\nu} F_{\rho\delta} = \frac{g^2}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}$ (*)2

- EXAMPLE 2

$$\mathcal{L} = \bar{\Psi}_R i \not{\partial} \Psi_R + \bar{\Psi}_L i \not{D} \Psi_L - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

In this theory only the left current couples to the gauge field

$\Rightarrow$ if we consider the Green function $\langle 0 | T J^L_\mu(z) J^L_\nu(x) J^L_\lambda(y) | 0 \rangle$, due to the symmetry, we loose gauge invariance!

$$\partial^\mu J^L_\mu = -\frac{g^2}{48\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \qquad (*)1$$

(*)1 More in detail, let us call the Green function of the three left current $\Gamma^L_{\mu\nu\lambda}(p,k)$

This function can be expressed as

$$\Gamma^L_{\mu\nu\lambda}(p,k) = \frac{1}{2}\Gamma_{\mu\nu\lambda}(p,k) - \frac{1}{2}\Gamma^S_{\mu\nu\lambda}(p,k)$$

(the factors $\frac{1-\gamma_5}{2}$ at the vertices can be commuted away

The first term is the Green function with 3 vector current $\Rightarrow$ no anomaly

We conclude that the anomaly of $\Gamma^L_{\mu\nu\lambda}(p,k)$ can be calculated by using the result for $\Gamma^S_{\mu\nu\lambda}$

But the Bose symmetry require symmetric conditions and we thus conclude that the left current is not conserved, thus breaking gauge invariance and the consistency of the theory.

(*)2 The anomalous Ward identity can be written as

$$i(p+k)^\mu \Gamma^S_{\mu\nu\lambda}(p,k) = -\frac{1}{2\pi^2}\varepsilon_{\mu\lambda\rho\nu}k^\mu p^\rho$$

multiply by photon polarizations $\varepsilon^\nu(p)\varepsilon^\lambda(k)$

$$i(p+k)^\mu\Gamma^S_{\mu\nu\lambda}(p,k) = -\frac{1}{2\pi^2}\varepsilon_{\mu\lambda\rho\nu}k^\mu p^\rho\,\varepsilon^\nu(p)\varepsilon^\lambda(k)$$

$$= \frac{1}{8\pi^2}\varepsilon_{\mu\lambda\rho\nu}\,i\left[p^\rho\varepsilon^\nu(p)-p^\nu\varepsilon^\rho(p)\right]i\left[k^\mu\varepsilon^\lambda(k)-k^\lambda\varepsilon^\mu(k)\right]$$

$$\sim F_{\mu\lambda}\tilde{F}^{\mu\lambda}$$

DIMENSIONAL REGULARIZATION

Perturbative computations in quantum field theory are affected by INFRARED (IR) and ULTRAVIOLET (UV) singularities

UV singularities are removed by the renormalization procedure, affect only loop integrals

IR singularities, associated to configurations in which loop momenta or external particles become soft and/or collinear to other particles, are more subtle. They cancel only after appropriate quantities (observables) are considered

A necessary procedure to deal with these divergences is to first REGULARIZE them.

A powerful technique to do this is to continue the number of dimensions to an arbitrary (complex) number $d \neq 4$

$g^{\mu\nu}$ d-dimensional metric — when $d=4$ $g^{\mu}{}_{\mu} = 4$

$\Rightarrow$ when $d \neq 4$ $g^{\mu}{}_{\mu} = d$

$$g^{\mu}{}_{\mu} = \underbrace{g_{\parallel}{}^{\mu}{}_{\mu}}_{4} + \underbrace{g_{\perp}{}^{\mu}{}_{\mu}}_{d-4}$$

$$g^{\mu\nu} = \underbrace{g^{\mu\nu}_{\parallel}}_{\text{in 4 dim}} + \underbrace{g^{\mu\nu}_{\perp}}_{\text{in } d-4}$$

When we deal with fermions we have to specify how the γ matrices and the Clifford algebra are extended in d dimensions

We can define as in $d=4$

$$\{\gamma^\mu, \gamma^\nu\} = 2 g^{\mu\nu} \mathbb{1}$$

$$\begin{cases} \gamma^{\mu\dagger} = \gamma^\mu & \text{if } \mu = 0 \\ \gamma^{\mu\dagger} = -\gamma^\mu & \mu \neq 0 \end{cases}$$

$$\mathrm{Tr}[\gamma^\mu \gamma^\nu] = g^{\mu\nu}\, \mathrm{Tr}\, \mathbb{1}$$

We must have $\mathrm{Tr}\,\mathbb{1} \to 4$ as $d \to 4$ $\Rightarrow$ We can conventionally set $\mathrm{Tr}\,\mathbb{1} = 4$ for all d

2)

How do we define γ^5 and $\varepsilon^{\mu\nu\rho\sigma}$?

In $d=4$ we have $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ $\quad (\varepsilon^{0123}=1)$

a) $\mathrm{Tr}[\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma] = i\,\mathrm{Tr}\,\mathbb{1}\,\varepsilon_{\mu\nu\rho\sigma}$

b) $\{\gamma_\mu, \gamma_5\} = 0$

In $d \neq 4$ it is not possible to preserve a) and b) (EXERCISE)

't Hooft-Veltman prescription

Define γ^5 as in 4 dimensions $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$

and that $\{\gamma^5, \gamma^\mu\} = 0$ if $\mu = 0,1,2,3$

$[\gamma^5, \gamma^\mu] = 0$ if $\mu \neq 0,1,2,3$

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} 1 & \text{if } \mu\nu\rho\sigma \text{ is an even permutation of } (0123) \\ -1 & \text{" " odd "} \\ 0 & \text{otherwise} \end{cases}$$