MMP I
Solution Sheet 10

HS 21
Prof. Ph. Jetzer

L. Buonocore, M. Loechner, X. Liu, M. Ebersold
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Exercise 1 [Geodesic on a cone (4 points)]

cone $z^2 = 8(x^2 + y^2) \rightarrow$ geodesic

cylindrical coordinates: $R = \sqrt{x^2 + y^2}$, $z = z$

$z^2 = 8R^2 \rightarrow z = \sqrt{8}R \rightarrow dz = \sqrt{8}dR$

$x = R\cos\theta \rightarrow dx = dR\cos\theta + R(-\sin\theta)d\theta$

$y = R\sin\theta \rightarrow dy = dR\sin\theta + R\cos\theta d\theta$

$$\begin{aligned} dx^2 + dy^2 &= dR^2\cos^2\theta + R^2\sin^2\theta d\theta^2 - 2R\,dR\,d\theta\cos\theta\sin\theta \\ &\quad + dR^2\sin^2\theta + R^2\cos^2\theta d\theta^2 + 2R\,dR\,d\theta\cos\theta\sin\theta \\ &= dR^2 + R^2 d\theta^2 \end{aligned}$$

$ds^2 = dx^2 + dy^2 + dz^2 = dR^2 + R^2d\theta^2 + 8dR^2 = 9dR^2 + R^2d\theta^2$

$$I = \int ds = \int \sqrt{9dR^2 + R^2d\theta^2} = \int \underbrace{\sqrt{9^2 + R^2\left(\frac{d\theta}{dR}\right)^2}}_{F(\theta',R)} dR, \qquad dR \neq 0$$

$F(\theta', R)$ doesn't depend on θ explicitly. $\rightarrow \frac{\partial F}{\partial \theta} = 0$

Euler Lagrange: $\frac{\partial F}{\partial \theta} - \frac{d}{dR}\left(\frac{\partial F}{\partial \theta'}\right) = 0 = \frac{d}{dR}\left(\frac{\partial F}{\partial \theta'}\right) \rightarrow \frac{\partial F}{\partial \theta'} = C$

$$\begin{aligned} \frac{\partial F}{\partial \theta'} &= \frac{R^2\theta'}{\sqrt{9 + R^2\theta'^2}} = C \\ \frac{R^2\theta'}{C} &= \sqrt{9 + R^2\theta'^2} \end{aligned}$$

$$
\begin{aligned}
R^4\theta'^2 &= c^2(9 + R^2\theta'^2) \\
\theta'^2(R^4 - c^2R^2) &= 9c^2 \\
\frac{d\theta}{dR} &= \sqrt{\frac{9c^2}{R^2(R^2-c^2)}} = \frac{3c}{R\sqrt{R^2-c^2}} \\
\theta &= \int \sqrt{\frac{9c^2}{R^2(R^2-c^2)}}dR = \int \frac{3c}{R\sqrt{R^2-c^2}}dR = 3\cos^{-1}\left(\frac{c}{R}\right) + A \\
\frac{\theta - A}{3} &= \cos^{-1}\left(\frac{c}{R}\right) \\
\rightarrow \cos\left(\frac{\theta - A}{3}\right) &= \frac{c}{R}
\end{aligned}
$$

Exercise 2 [Hermitian operators (4 points)]

A, B Hermitian operators (self-adjoint)

$$
\begin{aligned}
(Af, g) = (f, Ag) \leftrightarrow A = A^* \\
(Bf, g) = (f, Bg) \leftrightarrow B = B^*
\end{aligned}
$$

a) When is $C = AB$ Hermitian?

C is Hermitian if $(ABf, g) = (f, ABg)$

$$(ABf, g) = (Bf, A^*g) = (Bg, Ag) = (f, B^*Ag) = (f, BAg)$$

C is Hermitian $\leftrightarrow AB = BA \leftrightarrow [A, B] = 0$ (The commutator of A and B is 0)

b) For which values of α is C Hermitian?

1. $C = \alpha(AB + BA)$

We must show:

$$
\begin{aligned}
C \Rightarrow (\alpha(AB + BA)f, g) &= (f, Cg) = (f, \alpha(AB + BA)g)) \\
(Cf, g) = (\alpha(AB + BA)f, g) &= \alpha(ABf, g) + \alpha(BAf, g) \\
&= \alpha(f, BAg) + \alpha(f, ABg) = (f, \overline{\alpha}(AB + BA)g)
\end{aligned}
$$

C is Hermitian $\leftrightarrow \alpha = \overline{\alpha} \leftrightarrow \alpha \in \mathbb{R}$

2. $C = \alpha(AB - BA)$

It must be:

$$(Cf, g) = (f, Cg) = (f, \alpha(AB - BA)g)$$

$$
\begin{aligned}
(Cf,g) &= (\alpha(AB-BA)f,g) = \alpha(ABf,g) - \alpha(BAf,g) \\
&= \alpha(f,BAg) - \alpha(f,ABg) = -\alpha(f,(AB-BA)g) \\
&= (f,-\overline{\alpha}(AB-BA)g)
\end{aligned}
$$

C is Hermitian $\leftrightarrow \alpha = -\overline{\alpha} \leftrightarrow \alpha \in i\mathbb{R}$

3. $C = \alpha(AB + iBA)$

We must prove:

$$
\begin{aligned}
(Cf,g) &= (f,Cg) = (f,\alpha(AB+iBA)g) \\
(Cf,g) &= (\alpha(AB+iBA)f,g) = \alpha(f,BAg) + \alpha(f,-iABg) \\
&= \alpha(f,(BA-iAB)g) = (f,\overline{\alpha}(BA-iAB)g) \\
&= (f,-i\overline{\alpha}(AB+iBA)g)
\end{aligned}
$$

C is Hermitian $\leftrightarrow \alpha = -i\overline{\alpha}$

Exercise 3 [Eigenvalues and eigenspaces (4 points)]

P: linear operator in the space R of functions $f(x)$: complex and continuous on $[0,c]$

$$
Pf(x) = f(x) + k\int_0^c f(t)dt
$$

$\rightarrow$ Hermitian metric:

$$
(f,g) = \int_0^c f(x)\overline{g}(x)dx
$$

a) For which values of k is P Hermitian?

$$
(Pf,g) \underbrace{=}_{\text{must be}} (f,Pg) = \int_0^c dx f(x)\overline{\left(g(x) + k\int_0^c f(t)dt\right)}
$$

$$
\begin{aligned}
(Pf,g) &= \int_0^c Pf \cdot \overline{g}dx = \int_0^c \left(f + k\int_0^c fdt\right)\overline{g}dx \\
&= \int_0^c f\overline{g}dx + k\int_0^c f(t)dt\int_0^c \overline{g}(x)dx \\
&= \int_0^c f(x)\overline{g}(x)dx + \int_0^c f(t)dt\int_0^c \overline{(\overline{k}g(x))}dx \\
&= \int_0^c f(x)\overline{\left(g(x) + \overline{k}\int_0^c \overline{g}(t)dt\right)}\,dx
\end{aligned}
$$

P is Hermitian $\leftrightarrow k = \overline{k} \leftrightarrow k \in \mathbb{R}$

Eigenvalues and eigenspaces: $Pf_\lambda = \lambda f_\lambda$

$$Pf_\lambda = f_\lambda + k\int_0^c f_\lambda dt = \lambda f_\lambda$$

$$\underline{(\lambda - 1)f_\lambda = k\int_0^c f_\lambda dt}$$

If $\lambda = 1:$

$$k\int_0^c f_\lambda dt = 0 \rightarrow R_1 = \left\{ f \in R : \int_0^c f dt = 0 \right\}$$

If $\lambda \neq 1:$

$$f_\lambda = \frac{k}{\lambda - 1} \underbrace{\int_0^c f_\lambda dt}_{\text{const. (definite integral)}} = \text{const} =: A$$

A definite integral is just a number and since this equation is true for all eigenfunctions of λ its eigenspace is the one dimensional subspace of constant functions. We now determine the eigenvalue λ by plugging in the constant A, representing an eigenfuction:

$$A(\lambda - 1) = k\int_0^c f_\lambda dt = k\int_0^c A dt \rightarrow \cancel{A}(\lambda - 1) = kc\cancel{A}$$

$$\underline{\lambda = 1 + kc} \rightarrow R_{1+kc} = \{f_\lambda \in R : f_\lambda = \text{const}\}$$

b) Any Hermitian operator A for which $A^2 = A$ is a projecton operator (projector).

$$\begin{aligned} P^2 f(x) &= P\left[f(x) + k\int_0^c f(t)dt\right] \\ &= P\left[f(x)\right] + P\left[k\int_0^c f(t)dt\right] \\ &= P\left[f(x)\right] + k\int_0^c f(t)dt + k\underbrace{\int_0^c d\tilde{t}}_{c}\; k\int_0^c f(t)dt \\ &= Pf(x) + k\int_0^c f(t)dt\,(1 + kc) \\ &= Pf(x), \quad \forall f \quad \Leftrightarrow \quad k(1 + kc) = 0 \end{aligned}$$

Thus P is a projector for $k = 0$ and $kc + 1 = 0 \Leftrightarrow k = -\frac{1}{c}$.

Eigenvalues:

$$Pf = \lambda f \quad Pf = f(x) + k\int_0^c f(t)dt$$

$$\lambda = 1: \quad R_1 = \left\{ f \in R : \int_0^c f dt = 0 \right\}$$

$$\lambda = \underbrace{1 + kc}_{k = -1/c} = 0 \text{ with } R_0 = \{f \in R : f = \text{constant}\}$$