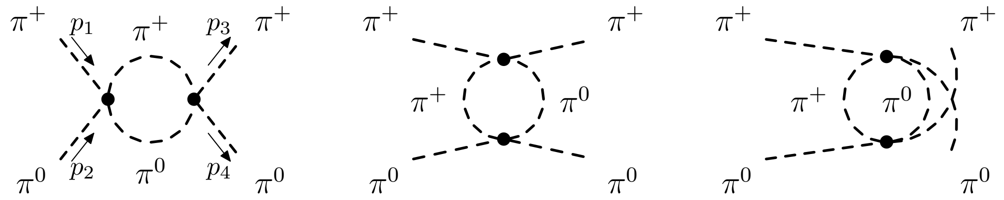

Figure 3.4: Diagrams contributing to $\pi-\pi$ scattering at one loop. The circle denotes a vertex from $\mathcal{L}_\chi^{(2)}$.

3.7 Loop corrections in χPT

χPT is a well-defined quantum field theory. As such, we must also consider quantum loops in this theory. Before discussing the most general case, it is illustrative to focus on the $\pi - \pi$ scattering amplitude at one-loop. There is a lot of information that can be obtained by a simple power counting analysis, without explicitly doing the computation. We can easily estimate the order in momenta of the corresponding loop graphs in Figure 3.4:

$$\mathcal{A}_{\text{loop}} \sim \int \frac{d^4p}{(2\pi)^4} \frac{p^2}{f^2} \frac{1}{p^4} \frac{p^2}{f^2} \sim \frac{p^2}{f^2} \times \frac{p^2}{(4\pi f)^2} \,. \tag{3.75}$$

Here p generically denotes any term that is of $\mathcal{O}(p_i, M_\pi)$, with p_i being the external momenta. The factor of $1/p^4$ is from the two internal propagators, and p^2/f^2 from the vertices. We see that the loop correction comes with a power suppression with respect to the tree level effect in (3.52), given by $p^2/(4\pi f)^2$. The additional factor of 4π is important because it gives some range to apply χPT to low energy processes involving pions and kaons. If the suppression factor were f instead of $4\pi f$, χPT would not be useful even for pions, since $M_\pi > f$. Note that there is an important difference here with respect to other quantum field theories: in χPT, loops are not suppressed by a small coupling but by powers of momentum over a hadronic scale. This is due to the fact that the chiral symmetry requires the pion couplings to vanish at zero momenta (and zero pion mass).

By simple power counting, it is also easy to see that the loop diagram is UV divergent. If we use a regularization that preserves the symmetries of the Lagrangian, such as dimensional regularization, the result of the loop calculation will necessarily be symmetric. Using dimensional regularization with $d = 4 - 2\epsilon$, we estimate the loop integral to give

$$\mathcal{A}_{\text{loop}} \sim \frac{p^4}{16\pi^2 f^4} \left[a \left(\frac{1}{\epsilon} + \log \frac{\mu^2}{p^2} \right) + b \right] \,, \tag{3.76}$$

with a, b being numerical constants that can be obtained from the explicit loop computation. Since, by construction, $\mathcal{L}_\chi$ contains all terms permitted by the chiral symmetry, all loop divergences can be absorbed by renormalizing the $\mathcal{L}_\chi$ couplings.[1]

[1] A mass-dependent regularization, such as a momentum cutoff, would explicitly break the chiral symmetry. As a result, we would obtain divergences that cannot be absorbed into a renormalization of the χPT couplings, making our loop computations more involved.

To absorb the divergence in (3.76), we need to add counterterms of $\mathcal{O}(p^4)$. Recalling our momentum expansion in (3.51), this means that we need to renormalize the couplings in $\mathcal{L}_\chi^{(4)}$. For our computation, in the $M_\pi = 0$ limit, we only need a subset of the $\mathcal{L}_\chi^{(4)}$ operators[2]

$$\begin{aligned}\frac{f^2}{4}\frac{1}{\Lambda_\chi^2}\mathcal{L}_\chi^{(4)} \supset L_1 \left\langle D_\mu\Sigma^\dagger D^\mu\Sigma\right\rangle^2 + L_2 \left\langle D_\mu\Sigma^\dagger D_\nu\Sigma\right\rangle \left\langle D^\mu\Sigma^\dagger D^\nu\Sigma\right\rangle \\ + L_3 \left\langle D_\mu\Sigma^\dagger D^\mu\Sigma\, D_\nu\Sigma^\dagger D^\nu\Sigma\right\rangle \,,\end{aligned} \tag{3.77}$$

which produce a four-pion interaction of $\mathcal{O}(p^4)$, when expanding Σ. The total $\mathcal{O}(p^4)$ $\pi - \pi$ scattering amplitude is a combination of tree graphs involving these interactions, and the loop contribution in (3.76). Schematically, we have

$$\mathcal{A}^{(4)} \sim \frac{p^4}{16\pi^2 f^4}\left[a\,\log\frac{\mu^2}{p^2} + b'\right] + \frac{4\,p^4}{f^4}L_i^r(\mu)\,, \tag{3.78}$$

written in terms of the renormalized couplings $L_i^r(\mu)$, which absorb the loop divergence, i.e.

$$L_i^r(\mu) = L_i(\mu) + \frac{f^2}{4}\frac{1}{16\pi^2 f^2}\left(a\,\frac{1}{\epsilon} + b - b'\right)\,. \tag{3.79}$$

The value of the constant $b - b'$ define a choice for the renormalization scheme. The total amplitude is μ independent. Therefore, in order to match the μ-dependence from the loop, $L_i^r(\mu)$ has to be of the form

$$L_i^r(\mu) = \frac{f^2}{4}\frac{1}{16\pi^2 f^2}\left[-a\,\log\frac{\mu^2}{\Lambda_\chi^2} + c_i\right]\,, \tag{3.80}$$

with c_i an undetermined numerical coefficient. This is the same as noting that the $L_i^r(\mu)$ couplings need to satisfy the (trivial) RGE equation

$$\frac{dL_i^r(\mu)}{d\log\mu} = -\frac{a}{32\pi^2}\,. \tag{3.81}$$

From (3.80), we see that a shift in the renormalization scale μ is compensated by a corresponding shift in $L_i^r(\mu)$, i.e.

$$L_i^r(\mu_2) = L_i^r(\mu_1) + \frac{f^2}{4}\frac{a}{(4\pi f)^2}\log\frac{\mu_1^2}{\mu_2^2}\,, \tag{3.82}$$

and so we expect $L_i^r(\mu)$ to be of the same order as these shifts. This naive dimensional analysis estimate let us infer the value of the hadronic scale Λ_χ in (3.77) in terms of the loop corrections

$$\Lambda_\chi \sim 4\pi f \approx 1\ \text{GeV}\,, \tag{3.83}$$

which is compatible with our initial estimate of $\Lambda_\chi \sim m_\rho$, based on general EFT arguments.

[2] The full $\mathcal{L}_\chi^{(4)}$ Lagrangian is presented in Section 3.8.

3.7.1 Weinberg's power counting theorem

We can generalize the features we observed in $\pi - \pi$ scattering to an arbitrary χPT amplitude. A generic χPT connected diagram gives the amplitude

$$\mathcal{A} \sim \int \left[\frac{d^4p}{(2\pi)^4}\right]^{N_L} \left[\frac{1}{p^2}\right]^{N_I} \prod_k \left(p^k\right)^{N_k} , \tag{3.84}$$

where N_L is the number of loops, N_I the number of internal lines, N_k the number of vertices of $\mathcal{O}(p^k)$ $(k = 2, 4, \dots)$, and p represents any term that is of $\mathcal{O}(p_i, M_j)$, with p_i and M_j being the external momenta and the light-meson masses, respectively. In a mass-independent regularization scheme, such as dimensional regularization, the only dimensional parameter is p. The amplitude must therefore have the following form $\mathcal{A} \sim p^D$, which defines its order in the chiral expansion. We can obtain the value of D by simple dimensional counting

$$D = 4N_L - 2N_I + \sum_k k\, N_k \,. \tag{3.85}$$

For any Feynman graph, one can show that

$$N_V - N_I + N_L = 1 \,, \tag{3.86}$$

where $N_V = \sum_k N_k$ is the total number vertices. This is a mathematical theorem in graph theory known as Euler's formula. Using Euler's formula, we can remove the number of internal lines from (3.85). By doing so, we obtain Weinberg's power counting formula [1]

$$D = 2 + 2N_L + \sum_k (k-2)\, N_k \,. \tag{3.87}$$

This expression connects the chiral counting in $\mathcal{L}_\chi$ and the loop counting into a single power counting in terms of powers of p. The chiral Lagrangian starts at $\mathcal{O}(p^2)$, so $k \geq 2$, and each term is non-negative. As a result, only a finite number of terms in $\mathcal{L}_\chi$ are needed to work at a given order in p, and the Lagragian is renormalizable order by order in the p expansion. The lowest order in the chiral expansion is $D = 2$. This is obtained for $N_L = 0$ and $k = 2$, which correspond to only tree-level graphs with $\mathcal{L}_\chi^{(2)}$ insertions. Since N_2 drops from (3.87), we can have an arbitrary number of $\mathcal{L}_\chi^{(2)}$ interactions, without increasing the order in p. The next order correction is at $\mathcal{O}(p^4)$. As we saw in the $\pi - \pi$ scattering example, at this order we have two possible contributions: tree-level graphs with one insertion of $\mathcal{L}_\chi^{(4)}$ ($N_L = 0$ and $N_4 = 1$), and loop graphs with an arbitrary number of $\mathcal{L}_\chi^{(2)}$ insertions ($N_L = 1$ and $N_{k>2} = 0$). One can proceed similarly for any order in the momentum expansion to find the different contributions.

Weinberg's power counting formula has an additional implication. Since each loop increases the chiral order, a generic χPT amplitude at $\mathcal{O}(p^4)$ has the same form as in (3.78), with a single logarithm. This is different to the SM case where, as we saw, there is an infinite series of logarithms that need to be resummed (see

i	$L_i^r(M_\rho)\times 10^3$	Source
1	0.4 ± 0.3	K_{e4}, $\pi\pi\to\pi\pi$
2	1.4 ± 0.3	K_{e4}, $\pi\pi\to\pi\pi$
3	-3.5 ± 1.1	K_{e4}, $\pi\pi\to\pi\pi$
4	-0.3 ± 0.5	Zweig rule
5	1.4 ± 0.5	$F_K : F_\pi$
6	-0.2 ± 0.3	Zweig rule
7	-0.4 ± 0.2	Gell-Mann–Okubo, L_5, L_8
8	0.9 ± 0.3	$M_{K^0}-M_{K^+}$, L_5, $(m_s-\hat{m}):(m_d-m_u)$
9	6.9 ± 0.7	$\langle r^2\rangle_V^\pi$
10	-5.5 ± 0.7	$\pi\to e\nu\gamma$

Table 3.1: Phenomenological values of the renormalized couplings $L_i^r(M_\rho)$. The last column shows the source used to extract this information. [Taken from [2]]

Section 1.5.2). Since in χPT there is just a single logarithm, which is commonly referred to as *chiral logarithm*, there is no need to do resummation, and one normally defines all the χPT $\mathcal{L}_\chi^{(4)}$ couplings at the same scale, typically $\mu = m_\rho$, $4\pi f_\pi$ or 1 GeV. The chiral logarithm is completely determined in terms of the lowest order terms in the chiral Lagrangian. On the other hand, the $\mathcal{L}_\chi^{(4)}$ couplings introduce additional unknown parameters. One can hope that the chiral logarithm is numerically more important than the $\mathcal{L}_\chi^{(4)}$ contributions, when one picks $\mu\sim 1$ GeV. This is formally correct, since $p^4\log(\Lambda_\chi^2/p^2) > p^4$. However, in practice $p\sim M_\pi$ or M_K, for which the logarithms give respectively 3.9 and 1.4. This is not a very large enhancement (especially for kaons). Still, the chiral logarithm provides useful estimates of the size of the $\mathcal{O}(p^4)$ corrections.

3.8 The $\mathcal{L}_\chi^{(4)}$ Lagrangian

In order to derive the $\mathcal{L}_\chi^{(4)}$ Lagrangian, one needs to list all possible operators of order $\mathcal{O}(p^4)$ invariant under parity, charge conjugation and local chiral symmetry. Although it is not strictly necessary, it is very convenient to reduce this list to a minimal basis of operators by using equations of motion, trace identities and integration by parts. The operator reduction procedure is similar to the one described in 2.3.1. The most general $\mathcal{L}_\chi^{(4)}$ Lagrangian reads [3]

$$\begin{aligned}
\mathcal{L}_4 =\, & L_1\,\langle D_\mu\Sigma^\dagger D^\mu\Sigma\rangle^2 \,+\, L_2\,\langle D_\mu\Sigma^\dagger D_\nu\Sigma\rangle\,\langle D^\mu\Sigma^\dagger D^\nu\Sigma\rangle \\
&+\, L_3\,\langle D_\mu\Sigma^\dagger D^\mu\Sigma D_\nu\Sigma^\dagger D^\nu\Sigma\rangle \,+\, L_4\,\langle D_\mu\Sigma^\dagger D^\mu\Sigma\rangle\,\langle\Sigma^\dagger\chi+\chi^\dagger\Sigma\rangle \\
&+\, L_5\,\langle D_\mu\Sigma^\dagger D^\mu\Sigma\left(\Sigma^\dagger\chi+\chi^\dagger\Sigma\right)\rangle \,+\, L_6\,\langle\Sigma^\dagger\chi+\chi^\dagger\Sigma\rangle^2 \\
&+\, L_7\,\langle\Sigma^\dagger\chi-\chi^\dagger\Sigma\rangle^2 \,+\, L_8\,\langle\chi^\dagger\Sigma\chi^\dagger\Sigma+\Sigma^\dagger\chi\Sigma^\dagger\chi\rangle \\
&-\, iL_9\,\langle F_R^{\mu\nu}D_\mu\Sigma D_\nu\Sigma^\dagger + F_L^{\mu\nu}D_\mu\Sigma^\dagger D_\nu\Sigma\rangle \,+\, L_{10}\,\langle\Sigma^\dagger F_R^{\mu\nu}\Sigma F_{L\mu\nu}\rangle \\
&+\, H_1\,\langle F_{R\mu\nu}F_R^{\mu\nu}+F_{L\mu\nu}F_L^{\mu\nu}\rangle \,+\, H_2\,\langle\chi^\dagger\chi\rangle\,.
\end{aligned} \tag{3.88}$$

As we see, the $\mathcal{O}(p^4)$ chiral Lagrangian introduce new couplings that are not determined by the chiral symmetry. These couplings parametrize our ignorance on

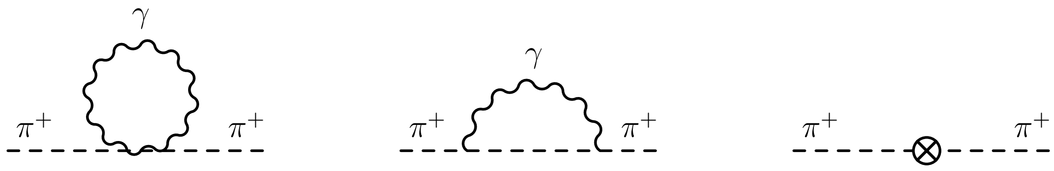

Figure 3.5: Diagrams contributing to the electromagnetic pion mass difference. Identical diagrams are obtained for the K^+.

the details of the underlying QCD dynamics. The operators with H_1 and H_2 couplings do not contain light-meson fields and cannot be measured directly. Therefore, we need to determine ten low-energy coefficients to specify the $\mathcal{O}(p^4)$ light-meson dynamics. In principle, these could be determined from QCD, for instance using lattice QCD. However, at present, the best determination of these couplings is obtained by fixing them from experimental measurements. As shown in Table 3.1, the experimental values for these coefficients are in reasonable agreement with the power counting estimate

$$L_i \sim \frac{f_\pi^2/4}{\Lambda_\chi^2} \sim \frac{1}{4(4\pi)^2} \sim 10^{-3} \,, \tag{3.89}$$

indicating a good convergence of the χPT momentum expansion below the resonance region, i.e. for $p < M_\rho$.

It seems reasonable to expect that the lowest-mass resonances, such as the ρ meson, should have an impact on the dynamics of the light-mesons. Interestingly, one can show that the values of the $\mathcal{L}_\chi^{(4)}$ couplings are mostly saturated by lowest-mass resonance exchange. For more details, the interested reader is encouraged to read the original article where this is shown [4].

3.9 QED corrections to the light-meson masses

As we saw in Section 3.6.1, we can use spurion analysis to compute the electromagnetic coupling to light mesons. We found that the $\mathcal{O}(p^2)$ QED couplings are given by

$$\begin{aligned} \mathcal{L}_\chi^{(2)} &\supset \frac{f^2}{4}[\, ie\, A_\mu \, \langle \partial^\mu \Sigma \, [Q, \Sigma^\dagger] \rangle + \text{h.c.}] + \frac{f^2}{4}\, e^2\, A_\mu A^\mu \, \langle [\Sigma^\dagger, Q]\, [Q, \Sigma] \rangle \\ &\supset [ie\, A_\mu \, (\pi^+ \partial^\mu \pi^- + K^+ \partial^\mu K^-) + \text{h.c.}] + e^2\, A_\mu A^\mu \, (\pi^+\pi^- + K^+K^-) \,, \end{aligned} \tag{3.90}$$

with $Q = \text{diag}(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})$. We can evaluate the electromagnetic correction to the light-meson masses by computing one-photon loops involving these interactions (see Figure 3.5). In addition to the loop contributions, we also need to include the local counterterms, in analogy to what we did in Section 3.7 in the $\pi - \pi$ scattering example. To match the loop contribution, the effective Lagrangian that contains the counterterm must read as follows

$$\begin{aligned} \mathcal{L}_{\text{em}}^{(2)} &= \frac{f^2}{4}\, e^2 \Delta^2 \, \langle [\Sigma^\dagger, Q]\, [Q, \Sigma] \rangle \\ &\supset e^2\, \Delta^2 \, (\pi^+\pi^- + K^+K^-) \,, \end{aligned} \tag{3.91}$$

with Δ being an undetermined coupling. Note that these electromagnetic corrections only affect the charged light-mesons, and that they are the same for π and for K. The loop diagrams in Figure 3.5 are zero in the massless meson limit, so the coupling Δ does not get renormalized and it is independent of the renormalization scale, i.e. $\Delta \neq \Delta(\mu)$. Neglecting terms of $\mathcal{O}(\alpha\, M^2_{\pi,K})$, the electromagnetic correction to the light-meson masses is entirely given by (3.91), and thus we find

$$\hat{M}^2_{\pi^+} = \hat{M}^2_{K^+} = e^2\Delta^2\,, \qquad\qquad \hat{M}^2_{\pi^0} = \hat{M}^2_{K^0} = \hat{M}^2_{\eta} = 0\,, \tag{3.92}$$

We can extract the value of Δ experimentally from the pion mass difference

$$M^2_{\pi^+} - M^2_{\pi^0} = e^2\Delta^2\,. \tag{3.93}$$

Using naive dimensional analysis, we can estimate the value of Δ to be

$$\Delta = \frac{\Lambda_\chi}{4\pi}\,C = \frac{4\pi f_\pi}{4\pi}\,C = f_\pi\, C = 92.4\,\mathrm{MeV} \times C\,, \tag{3.94}$$

with C of $\mathcal{O}(1)$. This is in good agreement with experimental value of the pion mass difference $M^2_{\pi^+} - M^2_{\pi^0} = 1.26 \times 10^3\,\mathrm{MeV}^2$, from where one extracts the value $\Delta = 117$ MeV.

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