MMP I
Tutorial 11

HS 2019
Prof. M. Grazzini

S. Devoto, M. Höfer, J. Yook
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Exercise 1: Linear operator in $\mathbb{C}^3$ (8 Pts.)

Consider the linear operator T that is represented by the hermitian matrix in the canonical basis:

$$T = \begin{pmatrix} 2 & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 2 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 2 \end{pmatrix}. \tag{1.1}$$

The matrix T can be diagonalised as $T = U^{-1} \cdot T_d \cdot U$, where T_d is the diagonal matrix with entries $\{3, 2, 1\}$.

a) Compute the trace and the determinant of T.

b) Find the spectral representation of T in the canonical basis, i.e. find the matrices of projection operators P_i in the canonical basis, such that

$$T = \sum_i E_i P_i. \tag{1.2}$$

c) Compute T^{99}, $e^T$and T^{-1}.

d) Compute the matrix R_T representing the resolvent $R_T(z) = (T - z)^{-1}$ in the canonical basis. For which values of z does the resolvent exist?

– please turn over –

Exercise 2: Perturbation theory (8 Pts.)
A linear operator $\mathcal{H} = \mathcal{H}_0 + V$ and a vector $|y\rangle$ in $\mathbb{C}^2$ are represented in the canonical basis by

$$\mathcal{H}_0 = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}, \quad V = \begin{pmatrix} \sin(a\gamma) & b_1\gamma^2 e^{b_2\gamma} \\ b_1\gamma^2 e^{b_2\gamma} & c_1\gamma^3 \end{pmatrix}, \quad |y\rangle = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \tag{2.1}$$

with $h_i, a, b_i, c_i, \gamma \in \mathbb{R}$. It often happens that the 'perturbation' V contains a small parameter γ. This allows for the construction of approximate solutions as an expansion in γ. To this end the resolvent is written as $R = R_0 \sum_{n=0}^{\infty}(-VR_0)^n$ with $R \equiv R_{\mathcal{H}}(z)$and $R_0 \equiv R_{\mathcal{H}_0}(z)$.

a) Solve $(\mathcal{H}_0 - z)|x\rangle = |y\rangle$ for $z \notin \{h_1, h_2\}$.

b) Solve $(\mathcal{H} - z)|x\rangle = |y\rangle$ approximately, taking into account all terms up to $\mathcal{O}(\gamma)$.

c) Solve $(\mathcal{H} - z)|x\rangle = |y\rangle$ approximately, taking into account all terms up to $\mathcal{O}(\gamma^2)$.

d) Show that the eigenvalues λ_i of $\mathcal{H}$ satisfy:

$$\lambda_i = h_i + \frac{\langle \phi_i^{(0)} | V | \phi_i \rangle}{\langle \phi_i^{(0)} | \phi_i \rangle} \tag{2.2}$$

where $|\phi_i^{(0)}\rangle$ and $|\phi_i\rangle$ are the eigenvectors of $\mathcal{H}_0$ and $\mathcal{H}$, respectively.

e) Compute approximately the eigenvalues of $\mathcal{H}$ taking into account all the terms up to $\mathcal{O}(\gamma)$.