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Applications of General Relativity

Solutions 8

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Exercise 1 [Hamiltonian geodesic formulation in PN framework]

a) The Hamiltonian for the full theory is

$$\mathcal{H} = \frac{1}{2}\left(-p_t^2\left(1-\frac{r_s}{r}\right)^{-1} + p_r^2\left(1-\frac{r_s}{r}\right) + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) . \tag{1}$$

b) This gives us the equation of motion:

$$\dot{t} = -p_t\left(1-\frac{r_s}{r}\right) , \tag{2}$$

$$\dot{r} = p_r\left(1-\frac{r_s}{r}\right) , \tag{3}$$

$$\dot{\theta} = \frac{p_\theta}{r^2} , \tag{4}$$

$$\dot{\phi} = \frac{p_\phi}{r^2\sin^2\theta} , \tag{5}$$

$$\dot{p}_t = 0 , \tag{6}$$

$$\dot{p}_r = \frac{r_s^2}{r^2}\left(-\frac{p_t^2}{2}\left(1-\frac{r_s}{r}\right)^{-2} - \frac{p_r^2}{2} + \frac{p_\theta^2}{rr_s^2} + \frac{p_\phi^2}{rr_s^2\sin^2\theta}\right) , \tag{7}$$

$$\dot{p}_\theta = \frac{p_\phi^2\cot\theta}{r^2\sin^2\theta} , \tag{8}$$

$$\dot{p}_\phi = 0 . \tag{9}$$

(c) Let's introduce a parameter ϵ to help us keep track of our orders. Let's choose $r_s = \mathcal{O}(\epsilon^2)$. For a massive particle we have $v \ll c$, so we find $p_t = \mathcal{O}(1)$, $p_r, p_\theta, p_\phi = \mathcal{O}(\epsilon)$, while for a massless particle we have $p_\mu = \mathcal{O}(1)$. To expand the EOM we e.g. make the replacements $r_s \to \epsilon^2 r_s, p_r \to \epsilon\, p_r, p_\theta \to \epsilon\, p_\theta, p_\phi \to \epsilon\, p_\phi$ and then expand to second order around $\epsilon = 0$.

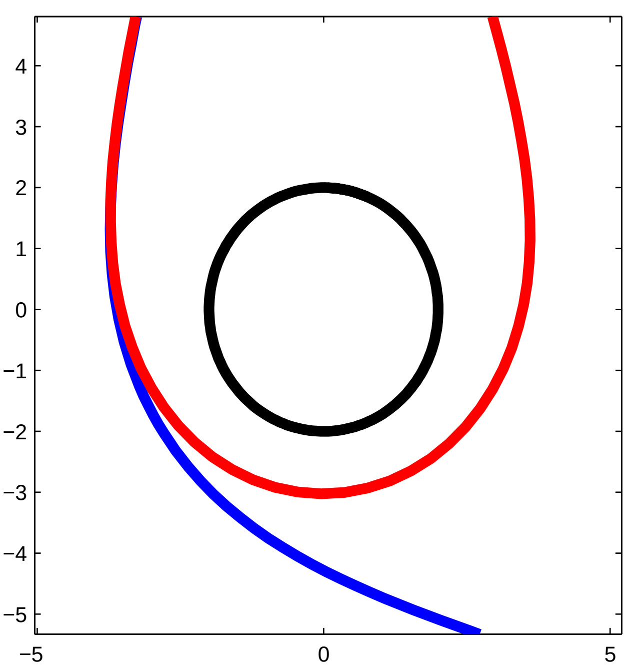

Figure 1: Two photons, each with the same impact parameter around a Schwarzschild black hole. Black shows the event horizon surface. Both photons have the same initial conditions, yet the red photon has been calculated using the full metric, and the blue the PN metric. Note that the red photon grazes the *photon sphere* at $r = 3r_s$.