CHAPTER 3

Chiral Perturbation Theory

All you really need to know for the moment is that the universe is a lot more complicated than you might think, even if you start from a position of thinking it's pretty damn complicated in the first place.

— *Douglas Adams*, The Hitchhiker's Guide to the Galaxy

In this part of the course we will introduce the basic ideas and methods of Chiral Perturbation Theory (χPT). This lecture summarizes some of the contents in [1–5], as well as Iain stewart's EFT course in `https://courses.edx.org/courses/MITx/8.EFTx/3T2014/course/`. The interested student is encouraged to follow these materials for a deeper discussion on the topics of the lecture, as well as for some other topics that will not be covered due to time limitations.

3.1 Introduction

It is well-established that Quantum Chromodynamics (QCD) is the theory describing the strong interactions. This theory is asymptotically free [6, 7] meaning that the strong coupling decreases as the energy increase, and perturbation theory can be applied at short distances. The resulting short-distance predictions from QCD have been tested and validated to a remarkable accuracy. However, the growing running of the QCD coupling make the theory non-perturbative at low energies, see Figure 5.1 (left). In the non-perturbative regime, the strong interactions lead to the confinement quarks and gluons, making very difficult to perform a QCD analysis in terms of these degrees of freedom. At such energies, a theoretical description in terms of composite quark and gluon states (hadrons) becomes more adequate.

The construction of such a theory is no simple task, given the richness of the hadronic spectrum [see Figure 5.1 (right)]. At very low-energies, below the resonance region ($E < M_\rho$), a great simplification takes place, and the hadronic spectrum reduces to only an octet of very light pseudo-scalar particles (π, K, η). The interactions of such light particles can be neatly described using χPT.

χPT is an effective theory (EFT) based on the global symmetry properties of QCD. This theory provides a clear example of bottom-up effective field theory, and will help us further explore other important theoretical concepts such as Goldstone bosons and non-linear symmetry representations. Furthermore, χPT provides an

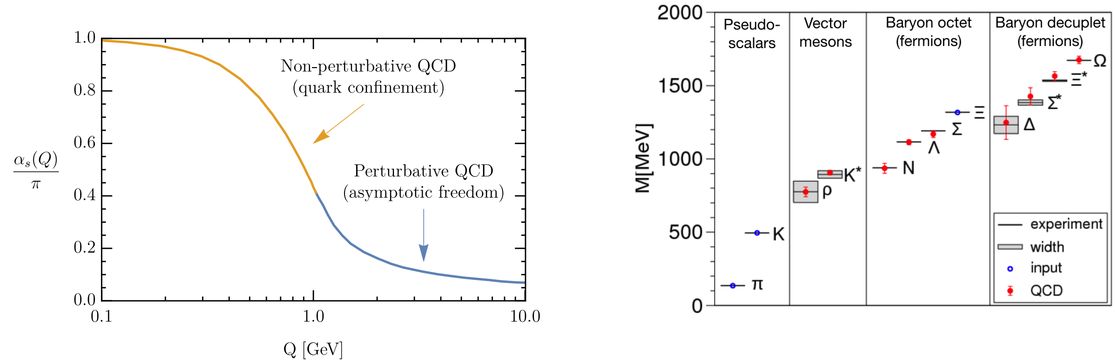

Figure 3.1: *Left:* Running of the QCD coupling. *Right:* The light hadron spectrum of QCD. Horizontal lines and bands correspond to the experimental values with their decay widths. The red dots denote lattice QCD predictions using the blue dots as input values. [Figure taken from [8]]

example of EFT where loops are not suppressed by a coupling constant, but instead they are suppressed by powers in a power expansion parameter. This different to what we saw in the previous chapters, where we were integrating out heavy particles, yielding contributions that are suppressed in powers of the coupling, but are the same order in the power expansion of the heavy scale Λ. So this will give us an example of a theory with a non-trivial power counting and we will allow us further develop the concept of power counting in EFTs.

3.2 The chiral symmetry in massless QCD

In the absence of quark masses and EW interactions, the QCD Lagrangian for N_f light-flavors arranged in a flavor multiplet $q_{L,R} = (q^1_{L,R}, q^2_{L,R}, \ldots, q^{N_f}_{L,R})^\intercal$ reads

$$\mathcal{L}^0_{\text{QCD}} = -\frac{1}{4}\, G^a_{\mu\nu}\, G^{\mu\nu\, a} + i\,\bar{q}_L \not{D}\, q_L + i\,\bar{q}_R \not{D}\, q_R\,, \tag{3.1}$$

with $D_\mu = \partial_\mu - ig_s T^\alpha G^\alpha_\mu$, with $T^\alpha\,(\alpha = 1, \ldots, 8)$ being the Gell-Mann matrices. In addition to $SU(3)_c$ local invariance, the QCD Lagrangian is invariant under the *global* symmetry in flavor space[1]

$$\mathcal{G}_{N_f} \equiv\; SU(N_f)_L \times SU(N_f)_R \times U(1)_V\,, \tag{3.2}$$

under which left- and right-handed transform as

$$q_L \overset{G}{\to} g_L\, q_L\,, \qquad q_R \overset{G}{\to} g_R\, q_R\,, \qquad g_{L,R} \in SU(N_f)_{L,R}\,. \tag{3.3}$$

[1]The $U(1)_V$ symmetry, which survives also in the case of non-vanishing masses, correspond to baryon number and it is trivially realized in the meson sector. The global $U(1)_A$, which is a limit of the classical QCD action, is explicitly broken at the quantum level. This is the so-called $U(1)_A$ anomaly. The discussion of this is beyond the scope of this lecture, but the interested reader is encouraged to explore [3], where the effects of the $U(1)_A$ anomaly in the context of χPT are discussed.

This global symmetry is commonly denoted as the QCD *chiral symmetry*. The chiral symmetry, which should be approximately good for the light quark sector, $q = (u, d, s)^\intercal$, is however not realized in the hadronic spectrum. Even though hadrons nicely fit into $SU(3)_V \equiv SU(3)_{L+R}$ multiplets, degenerate multiplets with opposite parity do not exist. Moreover, the octet of pseudo-scalar mesons is much lighter than the other hadronic states. This provides an strong indication that the ground state (vacuum) of the theory should not be symmetric under the chiral group. More specifically, it is expected that the symmetry $\mathcal{G}$ is spontaneously broken at low energies by the quark condensate[2]

$$\langle 0|\bar{q}_L^i q_R^j|0\rangle = \delta_{ij}\,\Lambda_\chi^3\,, \tag{3.4}$$

resulting in the Spontaneous Chiral Symmetry Breaking (SCSB) pattern

$$\mathcal{G} \equiv\ SU(N_f)_L \times SU(N_f)_R \times U(1)_V \stackrel{SCSB}{\longrightarrow} SU(N_f)_V \times U(1)_V\,. \tag{3.5}$$

According to the Goldstone theorem [9] (see below), the SCSB gives rise to a massless scalar field for each broken generator, referred as *Goldstone bosons* (GB). For $N_f = 3$, this corresponds precisely to eight GBs that we can identify with the eight lightest hadronic states (π^+, π^-, π^0, η, K^+, K^-, K^0 and $\bar{K}^0$). As we will see later, their small masses are generated by the small quark-masses, which explicitly break the chiral symmetry.

3.3 The Goldstone theorem

Goldstone theorem: If there is a continuous symmetry transformation under which the Lagrangian is invariant, then either the vacuum state is also invariant under the transformation, or there must exist spinless particles of zero mass or *Goldstone bosons*. The number of Goldstone Bosons corresponds to the number of continuous symmetries under which the vacuum is not invariant.

Proof: Consider a theory involving several scalar fields ϕ_i arranged into a multiplet ϕ,[3] and with a Lagrangian of the form

$$\mathcal{L} = (\text{terms with derivatives}) - V(\phi)\,. \tag{3.6}$$

The vacuum state of the theory is defined by the vacuum expectation values of ϕ, which we denote as $\langle\phi\rangle$, corresponding to the constant field configuration that minimizes the potential, i.e.

$$\left.\frac{\partial V(\phi)}{\partial \phi_i}\right|_{\phi=\langle\phi\rangle} = 0\,. \tag{3.7}$$

[2] Other operators with the quantum numbers of the vacuum, such as e.g. $\langle 0|G^a_{\mu\nu}\,G^{\mu\nu\,a}|0\rangle$, also form condensates. These do not play a relevant role in the spontaneous chiral symmetry breaking.

[3] The scalars ϕ could be either elementary fields or composite objects, e.g. arising from the confinement of fermion fields as in the QCD case.

The set of field configurations where $\phi = \langle\phi\rangle$ is known as the vacuum manifold. Expanding around this minimum we find

$$V(\phi) = V(\langle\phi\rangle) + \cancel{\left.\frac{\partial V(\phi)}{\partial\phi_i}\right|_{\phi=\langle\phi\rangle}}(\phi - \langle\phi\rangle)^i + \frac{1}{2}(\phi - \langle\phi\rangle)^i(\phi - \langle\phi\rangle)^j \left.\frac{\partial^2 V(\phi)}{\partial\phi_i\partial\phi_j}\right|_{\phi=\langle\phi\rangle} + \dots \tag{3.8}$$

Here the quadratic term

$$m_{ij} \equiv \left.\frac{\partial^2 V(\phi)}{\partial\phi_i\partial\phi_j}\right|_{\phi=\langle\phi\rangle}, \tag{3.9}$$

is a symmetric matrix whose eigenvalues give the mass of the scalars. These eigenvalues cannot be negative since $\langle\phi\rangle$ is a minimum. To prove the Goldstone theorem we need to show that every continuous symmetry transformation of the Lagragian that is not a symmetry of $\langle\phi\rangle$ yields a zero eigenvalue in m_{ij}.

Let us assume that the Lagrangian is invariant under a symmetry group $\mathcal{G}$ defined by the set of infinitesimal transformations

$$\delta\phi_i = \epsilon^\alpha\, T^\alpha_{ij}\, \phi_j\,. \tag{3.10}$$

where ϵ^α are infinitesimal parameters that define the transformation and T^α ($\alpha = 1, \dots, \dim(\mathcal{G})$) are the generators of $\mathcal{G}$. If the vacuum state were also invariant under these transformations, the vacuum expectation values of ϕ_i would satisfy[4]

$$T^\alpha_{ij}\, \langle\phi_j\rangle = 0\,. \tag{3.11}$$

We are going to examine the possibility that the vacuum is only invariant under a subgroup $\mathcal{H}$ of these transformations in which case

$$T^\alpha_{ij}\, \langle\phi_j\rangle \begin{cases} = 0\,, & \text{for } T^\alpha \in \mathcal{H} \\ \neq 0\,, & \text{for } T^\alpha \in \mathcal{G}/\mathcal{H} \end{cases}\,. \tag{3.12}$$

If we restrict to constant fields, the derivative terms in (5.19) vanish and the invariance of the Lagrangian under $\mathcal{G}$ implies that the potential alone must also be invariant. This invariance implies

$$\frac{\delta V(\phi)}{\delta\epsilon^\alpha} = 0 \Longrightarrow \frac{\partial V}{\partial\phi_i}\, T^\alpha_{ij}\, \phi_j = 0\,. \tag{3.13}$$

Differentiating with respect to ϕ_k and setting $\phi = \langle\phi\rangle$, we have

$$\left.\frac{\partial V}{\partial\phi_i}\right|_{\phi=\langle\phi\rangle} T^\alpha_{ik} + \left.\frac{\partial^2 V}{\partial\phi_k\partial\phi_i}\right|_{\phi=\langle\phi\rangle} T^\alpha_{ij}\, \langle\phi_j\rangle = 0\,. \tag{3.14}$$

[4] A vacuum is invariant under a general group transformation $g \in \mathcal{G}$ if $g\,\langle\phi\rangle = \langle\phi\rangle$. Under an infinitesimal group transformation this implies $(\delta_{ij} + i\epsilon^\alpha\, T^\alpha_{ij} + \dots)\,\langle\phi_j\rangle = \langle\phi_i\rangle \Leftrightarrow T^\alpha_{ij}\,\langle\phi_j\rangle = 0$, which is the condition we wrote. Similarly, if the vacuum is not invariant under $g \in \mathcal{G}$, we expect $T^\alpha_{ij}\,\langle\phi_j\rangle \neq 0$ for some α.

The first term vanishes since $\langle\phi\rangle$ is a minimum, yielding

$$m_{ki}\,T^{\alpha}_{ij}\,\langle\phi_j\rangle = 0\,. \tag{3.15}$$

If the transformation leaves the vacuum state unchanged, then from (5.11) we see that the relation above is trivial. However, if the vacuum is not invariant other the transformation, then (5.12) and the equation above imply that m_{ij} has a zero eigenvalue associated to the broken symmetry generators T^{α}. It is now clear that the number of massless particles is determined by the dimension of the coset $\mathcal{G}/\mathcal{H}$, or the number of spontaneously broken symmetries, which defines the vacuum manifold.

3.4 Linear and non-linear sigma models

Consider the so-called linear sigma model given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\cdot\partial^\mu\phi - \frac{\lambda}{4}\,(\phi\cdot\phi - v^2)^2\,, \tag{3.16}$$

where $\phi = (\phi_1,\ldots,\phi_N)^\intercal$ is a real N-component scalar field. This Lagrangian has a global $SO(N)$ symmetry where ϕ transforms as an $SO(N)$ vector. The potential has been chosen so that it is minimized at $|\,\langle\phi\rangle\,| = v$.[5] In this example, the vacuum manifold corresponds to the set of field configurations with $\phi_1^2+\phi_2^2+\cdots+\phi_N^2 = v^2$, which correspond to the $N-1$ dimensional sphere S^{N-1}. We can use the $SO(N)$ symmetry to rotate the vector $\langle\phi\rangle$ to a standard direction, which we fix to $(0,0,\ldots,v)$. The vacuum of the Lagrangian spontaneously break the $SO(N)$ symmetry to the $SO(N-1)$ subgroup acting on the first $N-1$ components. The group $SO(N)$ has $N(N-1)/2$ generators, so the number of GBs, which is equal to the number of broken generators, is: $N(N-1)/2-(N-1)(N-2)/2 = N-1$. The $N-1$ GBs correspond to rotations of the vector ϕ, which leave its length unchanged, i.e. to excitations along the vacuum manifold. The potential energy is unchanged under rotations of ϕ, so these modes are massless. The remaining mode is a radial excitation which changes the length of ϕ, and produces a massive excitation, with mass $\sqrt{2\lambda}\,v$.

The geometric interpretation of the Goldstone fields as excitations along the vacuum manifold S^{N-1} is not manifest in the Lagrangian of (5.16). A more clever (and simpler) way to capture this geometrical interpretation is obtained by using a exponential representation for the ϕ field[6]

$$\phi = \frac{1}{\sqrt{2}}\,(v+\rho)\,U(\pi)\,, \qquad\qquad U(\pi) = e^{i\sqrt{2}\,J^\alpha\,\pi^\alpha/v}\,, \tag{3.17}$$

where J^α ($\alpha = 1,\ldots,N-1$) are the $SO(N)$ broken generators, and ρ and π^a are a new basis for the N ϕ_i fields. The Lagrangian in terms of the new fields is

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\rho\,\partial^\mu\rho - \frac{\lambda}{4}\,(\rho^2+2\rho v)^2 + \frac{1}{2}\,(\rho+v)^2\,\langle\partial_\mu U^\intercal\,\partial^\mu U\rangle\,, \tag{3.18}$$

[5] Note that the constant field configuration $|\,\langle\phi\rangle\,| = 0$ correspond to a maximum of the potential.

[6] It is easy to show that this change of variables, which is only defined for small π^a, has unit Jacobean determinant. This warranties that this field redefinition do not modify the S-matrix elements, i.e. the physical predictions do not get affected by the field reparametrization.

where $\langle\cdot\rangle$ denotes the trace. In this new basis, it is clear that only ρ, which corresponds to the radial excitation, has a potential. As expected, the radial excitation gets a mass $m_\rho = \sqrt{2\lambda}\, v$ and the π fields remain massless. At small momenta compared to m_ρ, i.e. $p \ll m_\rho$, the ρ field decouples and can be integrated out. The Lagrangian reduces to

$$\mathcal{L} = \frac{1}{2}\, v^2 \, \langle \partial_\mu U^\intercal \, \partial^\mu U \rangle + \mathcal{O}\left(\frac{p^2}{m_\rho^2}\right) , \tag{3.19}$$

which describe universal (model-independent) self-interactions of the GB at very low energies. In order to be sensitive to the particular dynamical structure of the potential, and not just to its symmetry properties, one needs to test the model-dependent part involving the scalar field ρ (see exercise sheet 2).

From this example we can extract a few generic properties of GBs:

i) GBs are derivatively coupled. They describe the (local) orientation of the field ϕ in the vacuum manifold. A constant GB field would correspond to a configuration in which the field ϕ has been rotated by the same angle everywhere in spacetime, and corresponds to a vacuum that is equivalent to the standard vacuum, $\langle\phi\rangle = (0, 0, \ldots, v)$. Therefore, the Lagrangian must be independent of π^a when π^a is constant, so only gradients of π^a appear in the Lagrangian.

   Therefore, if we restrict ourselves to the low momentum regime, the theory is weakly coupled. This property is not clear from the Lagrangian in (5.16), and it implies that when computing processes with this Lagrangian one finds exact (and not very transparent) cancellations among different momentum-independent contributions (see Exercise sheet 2). The two functional forms of the Lagrangian should of course give the same physical predictions.

ii) The Goldstone Lagrangian is non-linear in the Goldstone fields, and it describes an infinite number of Goldstone self-interactions. GBs are constrained to live in the vacuum manifold, described by $\phi_1^2 + \phi_2^2 + \cdots + \phi_N^2 = v^2$ in our example, which is a non-linear constraint on the fields, hence the non-linear Lagrangian.

iii) The fields in the Lagrangian in (5.16) transform linearly under the $SO(N)$ symmetry, hence its name. On the other hand, the field ρ is a singlet of the global symmetry and the fields π^a transform non-linearly under $SO(N)$. That is why the representation of the field in (5.17) is said to be in a non-linear representation of the symmetry.

In general, non-linear symmetry realizations are the most effective way to describe GBs. The technology to do this was first worked out by Callan, Coleman, Wess and Zumino [10, 11], and therefore it is named after them as the CCWZ formalism.

3.5 The CCWZ formalism

The CCWZ formalism provides a general formalism to construct effective Lagrangians for spontaneously broken theories. Consider a theory with global symmetry group

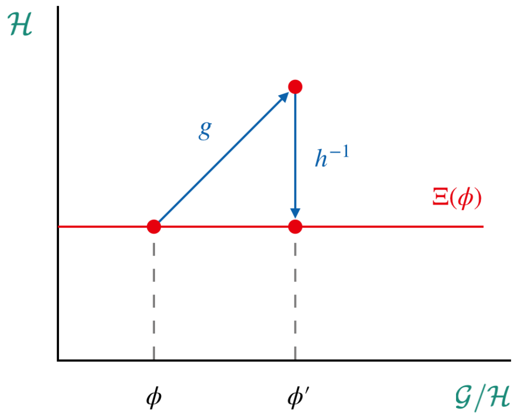

Figure 3.2: Under the action of $g \in \mathcal{G}$, $\Xi(\phi(x))$ transforms into some element of the ϕ' coset. A compensating $h(\phi, g) \in \mathcal{H}$ transformation is needed to go back to $\Xi(\phi')$.

$\mathcal{G}$ whose vacuum is only invariant under the subgroup $\mathcal{H}$, yielding the spontaneous symmetry breaking pattern $\mathcal{G} \to \mathcal{H}$. The vacuum manifold is given by the coset $\mathcal{G}/\mathcal{H}$. Vacuum manifolds are generically curved, as the S^{N-1} sphere in our example.

From the Goldstone theorem we know that the theory contains $N = \dim(\mathcal{G}) - \dim(\mathcal{H})$ GBs that correspond to excitations along the vacuum manifold. Our aim is to find a generic way to describe this set of GBs using only the symmetry information. We start by selecting a standard vacuum Φ_0 as the origin around which to do perturbation theory. We want to find the set of coordinates that describes small fluctuations in the vacuum manifold around the standard vacuum configuration. Let $\Xi(x) \in \mathcal{G}$ be an element that transform the standard vacuum configuration to the local field configuration. This element Ξ is not unique: $\Xi\, h$, with $h \in \mathcal{H}$, gives the same field configuration, since the standard vacuum is invariant under any transformation $h \in \mathcal{H}$, i.e. $h\,\Phi_0 = \Phi_0$. The CCWZ prescription consists in picking a set broken generators $\hat{T}^\alpha$, and choosing $\Xi(x)$ in what we denote as *canonical form*

$$\Xi(x) = \exp\left(i\,\hat{T}^\alpha\,\frac{\phi^\alpha(x)}{f}\right), \tag{3.20}$$

where ϕ^α are the GBs, and f is a normalization factor that makes the exponential dimensionless.

We want to determine how $\Xi(x)$ transforms under a global symmetry transformation $g \in \mathcal{G}$ (note that g is a global transformation, so it does not depend on x). The group element $\Xi(x)$ is transformed into a new element $g\,\Xi(x)$ that corresponds to a different point in the vacuum manifold. In general, the element $g\,\Xi(x)$ is not in its canonical form, but it can be written as[7]

$$g\,\Xi(x) = \Xi(x)'\,h(g, \Xi(x))\,, \tag{3.21}$$

with $h \in \mathcal{H}$ and where we made explicit the dependence of h on x trough its dependence on $\Xi(x)$. Choosing the canonical form for the transformed element, we

[7]Any element of the group $g \in \mathcal{G}$ can be written as $g = \Xi\, h$, where $\Xi \in \mathcal{G}/\mathcal{H}$ and $h \in \mathcal{H}$.

can therefore write the following transformation law for $\Xi(x)$ under $\mathcal{G}$

$$\Xi(x) \to \Xi(x)' = g\,\Xi(x)\,h(g,\Xi(x))^{-1}\,. \tag{3.22}$$

Figure 5.2 depicts the partition of the group elements (points in the plane) into cosets (vertical lines). Under a transformation $g \in \mathcal{G}$, the transformed element is not necessarily in its canonical form, and one needs a compensating transformation $h(g,\Xi) \in \mathcal{H}$ to bring it back to the canonical form. Note that the transformation in (5.32) is in general non-linear, except for when $g = h$, in which case it becomes linear. The CCWZ prescription provide the most general choice of non-linear representation which becomes linear when restricted to the unbroken subgroup. Any other choice gives the same results for all observables, but does not give the same off-shell Green functions.

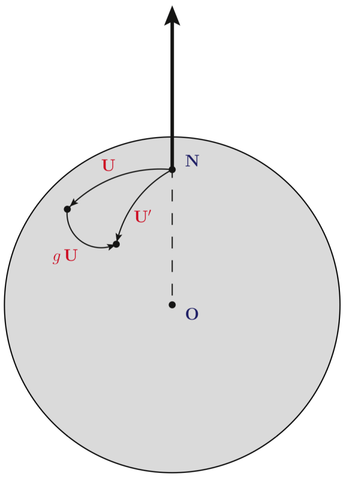

Figure 3.3: Geometrical representation of the S^2 vacuum manifold. The arrow indicates the chosen vacuum direction. [Figure taken from [1]]

Example 1: In the example of Section 5.4, $\mathcal{G} = SO(N)$, $\mathcal{H} = SO(N-1)$, and $\mathcal{G}/\mathcal{H} = S^{N-1}$. One can describe the orientation of the vector $\phi = (\phi_1, \ldots, \phi_N)^\intercal$ by giving the $SO(N)$ matrix $\Xi(x)$, where

$$\phi(x) = \Xi(x) \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v \end{pmatrix}\,. \tag{3.23}$$

The same configuration can also be described by $\Xi(x)\,h(x)$, where $h(x)$ is a matrix of the form

$$\begin{pmatrix} h'(x) & 0 \\ 0 & 1 \end{pmatrix}\,, \tag{3.24}$$

with $h'(x)$ an arbitrary $SO(N-1)$ matrix, since

$$\begin{pmatrix} h'(x) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ v \end{pmatrix}\,, \tag{3.25}$$

Let us take $N = 3$ for simplicity, so that the vacuum manifold is a sphere, which is easy to visualize. Figure 5.3 shows a geometrical representation of this vacuum manifold, with the north pole of the sphere indicating our choice for the standard vacuum. The unbroken symmetry group $\mathcal{H} = SO(2) = U(1)$ corresponds to rotations around the ON axis, where O is the center of the sphere. The group generators are J_1, J_2, J_3, where J_K generate the rotations along the kth axis, and the unbroken generator is J_3. The CCWZ prescription consists in choosing

$$\Xi(x) = \exp\left[\frac{i}{f}\left(J^1\pi^1(x) + J^2\pi^2(x)\right)\right]\,. \tag{3.26}$$

The matrix $\Xi(x)$ rotates a vector pointing along the z-axis to a different point in the sphere by rotating along longitude lines. Under a $g \in SO(3)$ transformation, the matrix $\Xi(x)$ is transformed to a new matrix $g\,\Xi(x)$ (see Figure 5.3)

$$g\,\Xi(x) = \Xi'(x)h(g,\Xi) \tag{3.27}$$

with $h \in SO(2)$. That h is non-trivial is a well-know property of rotations in three-dimensions. Take a vector and rotate it from A to B and then to C. This transformation is not the same as a direct rotation from A to C, but can be written as a rotation about OA, followed by a rotation from A to C. The transformation h is non-trivial the vacuum manifold $\mathcal{G}/\mathcal{H} = S^2$ is curved.

Example 2: For the second example, let us choose the chiral symmetry of QCD with N_f light quarks: $SU(N_f)_L \times SU(N_f)_R \to SU(N_f)_V$. The symmetry group is now $\mathcal{G} = SU(N_f)_L \times SU(N_f)_R$ and the unbroken subgroup $\mathcal{H} = SU(N_f)_V$. The vacuum manifold is $\mathcal{G}/\mathcal{H} = SU(N_f)_L \times SU(N_f)_R/SU(N_f)_V$, which is isomorphic to $SU(N_f)$. The generators of $\mathcal{G}$ are T^α_L and T^α_R ($\alpha = 1, 2, \ldots, N_f^2-1$), which act on left- and right- handed quarks respectively, and the generators of $\mathcal{H}$ are $T^\alpha_V = T^\alpha_L + T^\alpha_R$. The unbroken generators of $\mathcal{H}$ plus the broken generators $\hat{T}^\alpha$ span the space of all symmetry generators of $\mathcal{G}$, but the possible set of unbroken generators is not fixed, so we have to choose one. There are two commonly used bases for the QCD chiral Lagrangian, the ξ-basis and the Σ-basis, and we will consider them both. There are many simplifications that occur for QCD because the coset space $\mathcal{G}/\mathcal{H}$ is isomorphic to a Lie group. This is not true in general: as we saw, in the $SO(N)$ model, the coset space is S^{N-1} which is not isomorphic to a Lie group for $N \neq 4$.

1. The ξ basis.

One choice of broken generators is to pick $\hat{T}^\alpha_A = T^\alpha_L - T^\alpha_R$. Let a $SU(N_f)_L \times SU(N_f)_R$ transformation be represented in block diagonal form,

$$g = \begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix}, \tag{3.28}$$

where L and R are the $SU(N_f)_L$ and $SU(N_f)_R$ transformations, respectively. The unbroken transformation have the same form with $g_L = g_R = g_V$,

$$h = \begin{pmatrix} g_V & 0 \\ 0 & g_V \end{pmatrix}. \tag{3.29}$$

The Ξ field is then defined using the CCWZ prescription as

$$\Xi = e^{i\hat{T}^\alpha \pi^\alpha(x)/f_\xi} = \exp\left[\frac{i}{f_\xi}\begin{pmatrix} T^\alpha\pi^\alpha(x) & 0 \\ 0 & -T^\alpha\pi^\alpha(x) \end{pmatrix}\right] = \begin{pmatrix} \xi(x) & 0 \\ 0 & \xi(x)^\dagger \end{pmatrix}, \tag{3.30}$$

with $\xi(x) = \exp(i\,T^\alpha\pi^\alpha(x)/f_\xi)$. The transformation rule in (5.32) gives

$$\begin{pmatrix} \xi(x) & 0 \\ 0 & \xi(x)^\dagger \end{pmatrix} \to \begin{pmatrix} g_L & 0 \\ 0 & g_R \end{pmatrix}\begin{pmatrix} \xi(x) & 0 \\ 0 & \xi(x)^\dagger \end{pmatrix}\begin{pmatrix} g_V^{-1} & 0 \\ 0 & g_V^{-1} \end{pmatrix}. \tag{3.31}$$

This implies the transformation law for $\xi(x)$

$$\xi(x) \to g_L\, \xi(x)\, g_V^{-1}(x) = g_V(x)\, \xi(x)\, g_R^\dagger\,, \tag{3.32}$$

which defines g_V in terms of g_L, g_R and ξ.

2. The Σ basis.

The σ basis is obtained using the choice $\hat{T}^\alpha = T_L^\alpha$ for the broken generators. In this case the CCWZ prescription gives

$$\Xi = e^{i\hat{T}^\alpha \pi^\alpha(x)/f_\Sigma} = \exp\left[\frac{i}{f_\Sigma}\begin{pmatrix} T^\alpha\pi^\alpha(x) & 0 \\ 0 & 0\end{pmatrix}\right] = \begin{pmatrix}\Sigma(x) & 0 \\ 0 & 1\end{pmatrix}, \tag{3.33}$$

where $\Sigma(x) = \exp(i\, T^\alpha \pi^\alpha(x)/f_\Sigma)$. The transformation law gives

$$\begin{pmatrix}\Sigma(x) & 0 \\ 0 & 1\end{pmatrix} \to \begin{pmatrix} g_L & 0 \\ 0 & g_R\end{pmatrix}\begin{pmatrix}\Sigma(x) & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} g_V^{-1} & 0 \\ 0 & g_V^{-1}\end{pmatrix}. \tag{3.34}$$

which implies $g_V = g_R$, and

$$\Sigma(x) \to g_L\, \Sigma(x)\, g_R^\dagger\,. \tag{3.35}$$

Comparing with (5.32), we see that Σ and ξ are related by[8]

$$\Sigma(x) = \xi^2(x)\,. \tag{3.36}$$

and hence $f_\Sigma = f_\xi/2$.

It is interesting to check how the ξ and Σ fields transform under a parity transformation.[9] From the transformation properties derived in (5.32) and (5.36), we have that

$$\xi \xrightarrow{P} \xi^\dagger\,, \qquad\qquad \Sigma \xrightarrow{P} \Sigma^\dagger\,, \tag{3.38}$$

and therefore, for a Lagrangian in terms of these fields to be parity invariant, it needs to be invariant under the transformation $\Sigma \leftrightarrow \Sigma^\dagger$ ($\xi \leftrightarrow \xi^\dagger$). As we will show in the next section, this is the case for the χPT Lagrangian.

[8] Note that g_i ($i = V, L, R$) denote transformations under the $SU(N_f)_i$ group, and hence $g_i^{-1} = g_i^\dagger$.

[9] The generators of the chiral symmetry group transform under a parity transformation as

$$P: \qquad T_{L,R} \to T_{R,L}\,, \qquad T_V \to T_V\,, \qquad T_A \to -T_A\,, \tag{3.37}$$

and so $\mathcal{G} \xrightarrow{P} \mathcal{G}$, as expected since the QCD Lagrangian is P invariant.

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