1) The Lorentz group

The Lorentz group is the set of transformations a four-dimensional space time coordinates (x^μ), which preserve the scalar product $x \cdot y = x^\mu g_{\mu\nu} y^\nu$

$$x^\mu \to (x')^\mu = \Lambda^\mu{}_\nu x^\nu \qquad g_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$$
$$y^\mu \to (y')^\mu = \Lambda^\mu{}_\nu y^\nu$$

$$\Lambda^\mu{}_\lambda \Lambda^\nu{}_\rho \, g_{\mu\nu} = g_{\lambda\rho} \quad \Rightarrow \quad \boxed{\Lambda^T g \Lambda = g}$$

* Such transformations form a (non-compact) group
* The sub-set defined by $\det(\Lambda) = +1$, $\Lambda^0{}_0 \geq 1$ is denoted the proper Lorentz group
* 6 generators : $\begin{cases} J_i \leftrightarrow \text{rotations (3d)} \\ K_i \leftrightarrow \text{boosts} \end{cases}$

$$\Lambda^\mu{}_\nu = \left[\exp\left(-\tfrac{i}{2}\omega^{\rho\sigma} M_{\rho\sigma}\right)\right]^\mu{}_\nu$$

$$M^\mu{}_\nu = \begin{pmatrix} 0 & -K_1 & -K_2 & -K_3 \\ K_1 & 0 & J_3 & -J_2 \\ K_2 & -J_3 & 0 & J_1 \\ K_3 & J_2 & -J_1 & 0 \end{pmatrix}$$

$$[J_i, J_j] = i\epsilon_{ijk} J_k$$
$$[J_i, K_j] = i\epsilon_{ijk} K_k$$
$$[K_i, K_j] = -i\epsilon_{ijk} J_k$$

* We can re-arrange the generators into

$$J_i^\pm = \tfrac{1}{2}(J_i \pm iK_i) \quad \Rightarrow \quad \begin{cases} [J_i^\pm, J_j^\pm] = i\epsilon_{ijk} J_k^\pm \\ [J_i^\pm, J_j^\mp] = \emptyset \end{cases}$$

$\Rightarrow$ The Lorentz Group is "similar" to a "pair" of $SU(2)$ groups

More precisely
$$SO(4) \sim SU(2) \times SU(2)$$
$$SO(1,3) \sim SU(2) \times SU(2)^*$$

Lorentz

- charge conjugation interchange the two $SU(2)$'s $\left((J_i^{\pm})^* = -J_i^{\mp} \right)$
- the representations are <u>not</u> real (as in $SO(4)$)

This fact allows us to establish a further group equivalence, namely

$$SO(1,3) \sim SL(2,\mathbb{C})$$

<u>Group of 2×2 matrices with unit determinant</u>

Actually there is an <u>homomorphism</u> between $SO(1,3)$ & $SL(2,\mathbb{C})$

$\Rightarrow$ for any $A, B \in SL(2,\mathbb{C})$, for any representation of the Lorentz group, there exists $\Lambda(A)$:

$$\Lambda(A)\,\Lambda(B) = \Lambda(A \cdot B)$$

<u>Proof</u>: Let's consider the matrices

$$\sigma^\mu = (\sigma_0, -\sigma_i) \qquad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\sigma_i =$ Pauli matrices

A) Let's consider the following mapping

$$\rho : x^\mu \longrightarrow X = x_\mu \cdot \sigma^\mu \qquad (\rho : M_4 \to H)$$

general complex 2x2 matrix that scan the full set of 2x2 Hermitian complex matr.

(σ^μ = complete set)

$$\det X = \det(x_\mu \sigma^\mu x_\nu \sigma^\nu) = x_\mu x_\nu \underbrace{\det(\sigma^\mu \sigma^\nu)}_{= g^{\mu\nu}} = x_\mu x^\mu$$

B) Let's consider the following additional mapping

$$a : X \to X' = \underset{SL(2,\mathbb{C})}{A} X A^+$$

$$\det(X) = \det(X') \quad \& \quad (X')^+ = X' \Rightarrow H \xrightarrow{A} H$$

C) Let's finally consider

$$\rho^{-1} : Y \longrightarrow y^\mu = \frac{1}{2} \mathrm{Tr}(Y \cdot \bar{\sigma}^\mu) \qquad (\rho^{-1} : H \to M_4)$$

same as σ^μ

$$\rho^{-1} : X = x_\mu \sigma^\mu \longrightarrow \frac{1}{2} \mathrm{Tr}(x_\nu \sigma^\nu \bar{\sigma}^\mu) = \frac{1}{2} x_\nu \underbrace{\mathrm{tr}(\sigma^\nu \bar{\sigma}^\mu)}_{= 2 g^{\nu\mu}} = x^\mu$$

$$\Downarrow$$

$$M_4 \longrightarrow H \longrightarrow H \longrightarrow M_4$$

$$x_\nu \longrightarrow x_\nu \sigma^\nu \longrightarrow A x_\nu \sigma^\nu A^+ \longrightarrow \frac{1}{2} \mathrm{tr}(A x_\nu \sigma^\nu A^+ \bar{\sigma}^\mu) = x'^\mu$$

$$\Downarrow$$

$$\Lambda^\mu{}_\nu(A) = \frac{1}{2} \mathrm{Tr}(\bar{\sigma}^\mu A \sigma_\nu A')$$

The relation $\Lambda^{\mu}{}_{\nu}(M)$ $\qquad M \in SL(2,\mathbb{C})$

can be inverted up to a sign ambiguity

$$M(\Lambda) = \pm \frac{1}{\left[\det\left(\Lambda^{\mu}{}_{\nu}\,\sigma_{\mu}\bar{\sigma}^{\nu}\right)\right]^{\frac{1}{2}}} \Lambda^{\mu}{}_{\nu}\,\sigma_{\mu}\bar{\sigma}^{\nu}$$

$$\Lambda \longrightarrow \pm M$$

This correspondence defines a 2-value representation of the Lorentz group that is nothing but the <u>spinor representation</u>

Technically $\qquad SO(1,3)\Big|_{\substack{\text{proper}\\ \text{transf.}}} \longleftrightarrow SL(2,\mathbb{C})/\mathbb{Z}_2$