Applications of General Relativity

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Prof. P. Jetzer

Solutions 5

S. Balmelli, R. Bondarescu, D. Fiacconi
Website: http://www.physik.uzh.ch/lectures/agr/

Exercise 1 [More on the linearized EFE] (4 points)

$$\gamma_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h^{\sigma}{}_{\sigma}. \tag{1}$$

Let's start from the linearized Riemann tensor:

$$R_{\alpha\mu\beta\nu} = \frac{1}{2}\left(h_{\alpha\nu,\mu\beta} + h_{\mu\beta,\nu\alpha} - h_{\mu\nu,\alpha\beta} - h_{\alpha\beta,\mu\nu}\right) \tag{2}$$

Contracting, we obtain the Ricci tensor,

$$R_{\mu\nu} = \frac{1}{2}\left(h_{\nu\alpha}{}^{;\alpha}{}_{,\eta} + h_{\mu\alpha}{}^{;\alpha}{}_{,\nu} - \Box h_{\mu\nu} - h_{,\mu,\nu}\right). \tag{3}$$

Now in the Lorentz gauge, $\gamma_{\mu\nu}{}^{;\nu} = 0$, which implies $h_{\mu\nu}{}^{;\nu} = \frac{1}{2}h_{,\mu}$. So therefore

$$R_{\mu\nu} = \frac{1}{2}\left(h_{,\mu,\nu} - \Box h_{\mu\nu} - h_{,\mu,\nu}\right) = -\frac{1}{2}\Box h_{\mu\nu}. \tag{4}$$

And so the field equations become

$$R_{\mu\nu} = \frac{1}{2}g_{\mu\nu}R = -\frac{1}{2}\left(\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h\right) = -\frac{1}{2}\Box\gamma_{\mu\nu}, \tag{5}$$

so that

$$\Box\gamma_{\mu\nu} = -16\pi T_{\mu\nu}. \tag{6}$$

Exercise 2 [Asymptotic Justice] (8 points)

a) The point particle falls freely and follows a radial geodesic, i.e. a geodesic satisfying $\dot\theta = \dot\phi = 0$, where dot denotes derivation with respect to the affine parameter λ. Indeed, a geodesic starting from $x^{\nu}_{*} = (t_*, r_*, \theta_*, \phi_*)$ with vanishing initial velocity $\dot x^i(x^{\nu}_{*}) = 0$ will not have any reason to prefer any direction for θ and ϕ, due to the spherical symmetry.

The Lagrangian can be written as:

$$L = g_{\mu\nu}\dot x^{\mu}\dot x^{\nu} = \Delta(r)\dot t^2 - \frac{1}{\Delta(r)}\dot r^2, \quad \Delta(r) = 1 - \frac{2GM}{r}. \tag{7}$$

As usually, being the coordinate t cyclic, we find $\Delta\dot t = const$. Now, if we choose a proper-time parametrization of the coordinates, and considering that $\frac{dt}{d\tau} = \frac{1}{\sqrt{\Delta(r_*)}}$ at the moment when the body is released, we can conclude that

$$\Delta(r)\frac{dt}{d\tau} = \sqrt{\Delta(r_*)}. \tag{8}$$

Inserting this into the Lagrangian, and with the additional on-shell condition $g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 1$, we obtain the equation

$$\frac{1}{\Delta(r)}\left(\Delta(r_*) - \left(\frac{dr}{d\tau}\right)^2\right) = 1. \tag{9}$$

As a last step, the velocity can be obtained in terms of the time coordinate through

$$\frac{dr}{dt} = \frac{dr}{d\tau}\left(\frac{dt}{d\tau}\right)^{-1} = -\Delta(r)\sqrt{1 - \frac{\Delta(r)}{\Delta(r_*)}}.$$

As $\Delta(r) \to 0$ as $r \to 2GM$, the coordinate speed tends to zero as the body reaches the Schwartzschild radius. We can do a lower-bound estimation for the total coordinate time t_{hor} needed to reach the horizon as follows:

$$t_{hor} = \int_{r_*}^{2GM} \frac{dt}{dr}dr \geq \int_{2GM}^{r_*} \frac{r}{r - 2GM}dr \geq 2GM\int_{2GM}^{r_*} \frac{1}{r - 2GM}dr = +\infty.$$

The external observer thus sees the body asymptotically approaching the Schwartzschild radius, but never crossing it.

b) The equation for geodesic deviation is:

$$\frac{d^2}{d\tau^2}\delta x^\mu = R^\mu_{\nu\rho\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\delta x^\sigma$$

where δx^μ is the difference between the two geodesics we are looking at, and $x^\mu(\tau)$ is the parametrisation of one of the geodesics (as they are close, they are equal at leading order).

As the physicist is falling radially, we have $\delta x^\mu = l\delta^\mu_r$, where l is its length. As it is on a radial geodesic, we have $\dot{x}^\theta = \dot{x}^\phi = 0$. We are looking for the tidal forces, i.e. the difference in gravitational force felt by the two ends of the physicist: $\mu\frac{d^2}{d\tau^2}\delta x^r$. This is given by the geodesic deviation equation, as long as we can assume that the two ends lie on the same geodesic, i.e. $l \ll M$. We can assume that the object does not deform the Schwarzschild metric as long as $\mu \ll M$. Since we have to find a numerical value, we restore the use of c in the equations.

The geodesic deviation equation becomes:

$$\frac{d^2}{d\tau^2}\delta x^r = lR^r{}_{\nu\rho r}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}$$

wih $\nu, \rho \in \{t, r\}$. The only non-vanishing such Riemann tensor component in the Schwartzschild metric is:

$$R^r{}_{ttr} = \partial_t\Gamma^r{}_{tr} - \partial_r\Gamma^r{}_{tt} + \Gamma^\alpha_{tr}\Gamma^r{}_{\alpha t} - \Gamma^\alpha_{tt}\Gamma^r{}_{\alpha r} = -\frac{1}{2}\Delta(r)\Delta''(r)$$

We already have from exercise 1 a) the form of $\frac{dt}{d\tau}$ for a radial geodesic parametrised in proper time:

$$\frac{dt}{d\tau} = \frac{\sqrt{\Delta(r_*)}}{\Delta(r)}$$

This yields:

$$\frac{d^2}{d\tau^2}\delta x^r = -\frac{l}{2}\frac{\Delta(r_*)}{\Delta(r)}\Delta''(r)c^2.$$

The tidal force will thus be:

$$F(r) = \mu\frac{d^2}{d\tau^2}\delta x^r = -\frac{\mu l}{2}\frac{\Delta(r_*)}{\Delta(r)}\Delta''(r) = \mu l \Delta(r_*)\frac{2GMc^2}{r^2(rc^2 - 2GM)}$$

Inserting the numerical values and solving for r, we find that the astrophysicist will break at:

$$R \approx 840 \cdot r_S.$$