MMP I
Tutorial 13

HS 2017
Prof. M. Grazzini

S. Devoto, J. Yook, J. Mazzitelli

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http://www.physik.uzh.ch/en/teaching/PHY312/HS2017.html

Exercise 1: Fourier series (4 Pts.)

Let $\tilde{f} : (-\pi, \pi] \to \mathbb{R}$ be the function defined as $\tilde{f}(x) = |x|$. Consider its periodic extension $f : \mathbb{R} \to \mathbb{R}$.

a) Find the Fourier series of $f(x)$ and discuss its convergence.

b) Use the above results to determine $\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$.

Exercise 2: Variation calculus (6 Pts.)

a) Consider the point $P = (x_0, y_0)$ in the plane and the straight line r given by the equation $x = x_1$. Use variation calculus to show that the line which minimises the distance between P and r is the orthogonal segment from P to r.

b) Dido's problem: given a rope with fixed length l and a line segment with length $d < l$, find the shape of the rope such that the area enclosed by the rope and the segment is maximal.

Exercise 3: Differential equations (5 Pts.)

Solve the following differential equations

a) $y' = e^x y \ln y$

b) $x^2 y' = x^2 y^2 + xy + 1$

c) $y'' + 3y' - 10y = 3e^x$

d) $y' = (9x^2 - 2xy)/(2y + x^2 + 1)$

Exercise 4: Abstract spaces (4 Pts.)

Consider the vector space $H = C[a, b]$ of the continuous functions $f : [a, b] \to \mathbb{R}$ with the scalar product

$$\langle x|y\rangle = \int_a^b x(t)y(t)\mathrm{d}t$$

Explain why H is not a Hilbert space.

– please turn over –

Exercise 5: Linear operators (5 Pts.)

Consider the Hilbert space $H = \mathbb{C}^n$. Its elements are the n-dimensional vectors $x = (x_1, x_2..., x_n)$ with $x_i \in \mathbb{C}$ and $n \in \mathbb{N}$. The scalar product is defined as

$$\langle x|y\rangle = \sum_{i=1}^{n} x_i^* y_i$$

Show that every linear operator $T : H \to H$ is bounded.

Exercise 6: Differential equation systems (6 Pts.)

Solve the following differential equation systems (write the solutions in their real representations in all cases):

$$a) \begin{cases} \dot{x} = 3x + 4y - 2z \\ \dot{y} = 2x + y - 4z \\ \dot{z} = x + 2y \end{cases} \qquad b) \begin{cases} \dot{x} = 2x - 2y - z \\ \dot{y} = 3x - 5y - 3z \\ \dot{z} = 2x - 4y - z \end{cases} \qquad x(0) = 2, y(0) = -2, z(0) = -1$$

Exercise 7: Orthonormal bases (5 Pts.)

Consider the Hilbert space $H = L^2\,[0, \infty]$. Define the vectors $|x_0\rangle = \exp\{-t/2\}$, $|x_i\rangle = t\,|x_{i-1}\rangle$. Show that the $|x_i\rangle$ are linearly independent. Argue that $\text{span}\{|x_i\rangle\}$ is dense in H. Use the Gram-Schmidt procedure to build an orthonormal basis. Explicitly construct the first three orthonormal basis vectors.

Exercise 8: Linear operator in $\mathbb{C}^3$ (8 Pts.)
Consider the linear operator T that is represented by the hermitian matrix in the canonical basis:

$$T = \begin{pmatrix} 2 & -\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 2 & 0 \\ -\frac{i}{\sqrt{2}} & 0 & 2 \end{pmatrix}. \tag{8.1}$$

The matrix T can be diagonalised as $T = U^{-1} \cdot T_d \cdot U$, where T_d is the diagonal matrix with entries $\{3, 2, 1\}$.

a) Compute the trace and the determinant of T.

b) Find the spectral representation of T in the canonical basis, i.e. find the matrices of projection operators P_i in the canonical basis, such that

$$T = \sum_i E_i P_i. \tag{8.2}$$

c) Compute T^{99}, $e^T$and T^{-1}.

d) Compute the matrix R_T representing the resolvent $R_T(z) = (T - z)^{-1}$ in the canonical basis. For which values of z does the resolvent exist?

Exercise 9: Laplace operator on a circle (4 Pts.)
Consider the two-dimensional Laplace equation $\nabla^2\psi(r, \phi) = 0$ and let C_R be the circle of radius R centred at the origin.

a) Find the solution to $\nabla^2\psi = 0$ that is regular inside C_R and satisfies the Dirichlet boundary condition $\psi(R, \phi) = \sin^2(2\phi)$ on C_R.

b) Find the solution to $\nabla^2\phi = 0$ that is regular outside C_R and satisfies the Dirichlet boundary condition $\psi(R, \phi) = \sin^2(2\phi)$ on C_R.