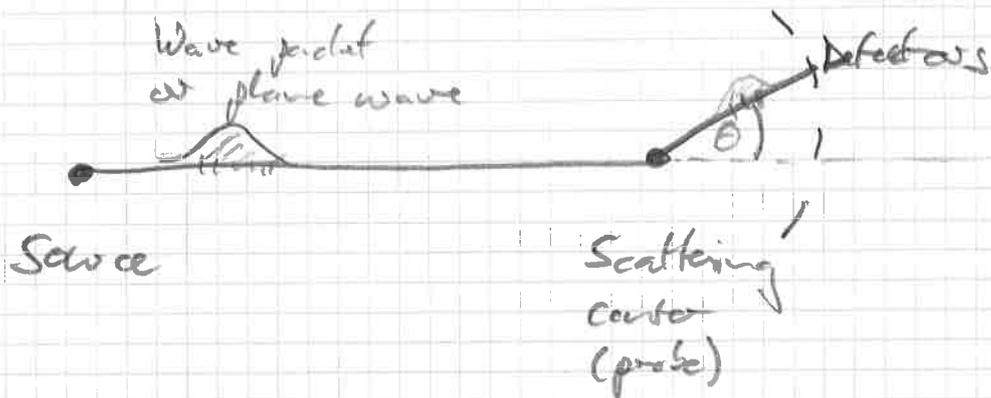


Scattering theory



- scattering center: potential $V(\vec{x})$
- incoming and outgoing waves: energy eigenstates (continuum) of $H = H_0 + V$

Lippmann-Schwinger equation

time-independent scattering process

- continuous source
- classical time-independent potential

Starting point:

$$H_0 = \frac{\vec{p}^2}{2m} = -\frac{\hbar^2 \Delta}{2m} \quad ; \quad H = H_0 + V$$

- spectrum of H_0 : $|\vec{p}\rangle$ = momentum eigenstates
- elastic scattering \rightarrow no change in energy
 \rightarrow same energy eigenstates

eigenvalue equation for $V=0$

$$H_0 |\phi\rangle = E |\phi\rangle$$

$|\phi\rangle$: plane or spherical wave

eigenvalue equation for scattering

$$(H_0 + V) |\gamma\rangle = E |\gamma\rangle \quad \text{with: } |\gamma\rangle \xrightarrow{V \rightarrow 0} |\phi\rangle$$

for E fixed

formal solution (Lippmann-Schwinger)

$$|\gamma^\pm\rangle = |\phi\rangle + \frac{1}{E - H_0 \pm i\epsilon} V |\gamma^\pm\rangle$$

$\epsilon > 0$: infinitesimal displacement of E in complex plane: to keep $E - H_0$ well-defined

Solution in position space:

$$\langle \vec{x} | \gamma^\pm \rangle = \langle \vec{x} | \phi \rangle + \int d^3x' \langle \vec{x} | \frac{1}{E - H_0 \pm i\epsilon} | \vec{x}' \rangle \langle \vec{x}' | V | \gamma^\pm \rangle$$

Integral equation with kernel

$$G_\pm(\vec{x}, \vec{x}') = \frac{\hbar^2}{2m} \langle \vec{x} | \frac{1}{E - H_0 \pm i\epsilon} | \vec{x}' \rangle$$

$$= -\frac{1}{4\pi} \frac{e^{\pm i k |\vec{x} - \vec{x}'|}}{|\vec{x} - \vec{x}'|} \quad \text{with } k = \frac{1}{\hbar} \sqrt{2mE}$$

is Green's function of the Helmholtz equation

$$(\Delta + k^2) G_{\pm}(\vec{x}, \vec{x}') = \delta^3(\vec{x} - \vec{x}')$$

such that

$$\langle \vec{x} | \gamma_{\pm} \rangle = \langle \vec{x} | \phi \rangle - \frac{2m}{\hbar^2} \int d^3x' \frac{e^{\pm ik|\vec{x} - \vec{x}'|}}{4\pi|\vec{x} - \vec{x}'|} \langle \vec{x}' | V | \gamma_{\pm} \rangle$$

↑
incoming wave
& no scattering

scattering contribution

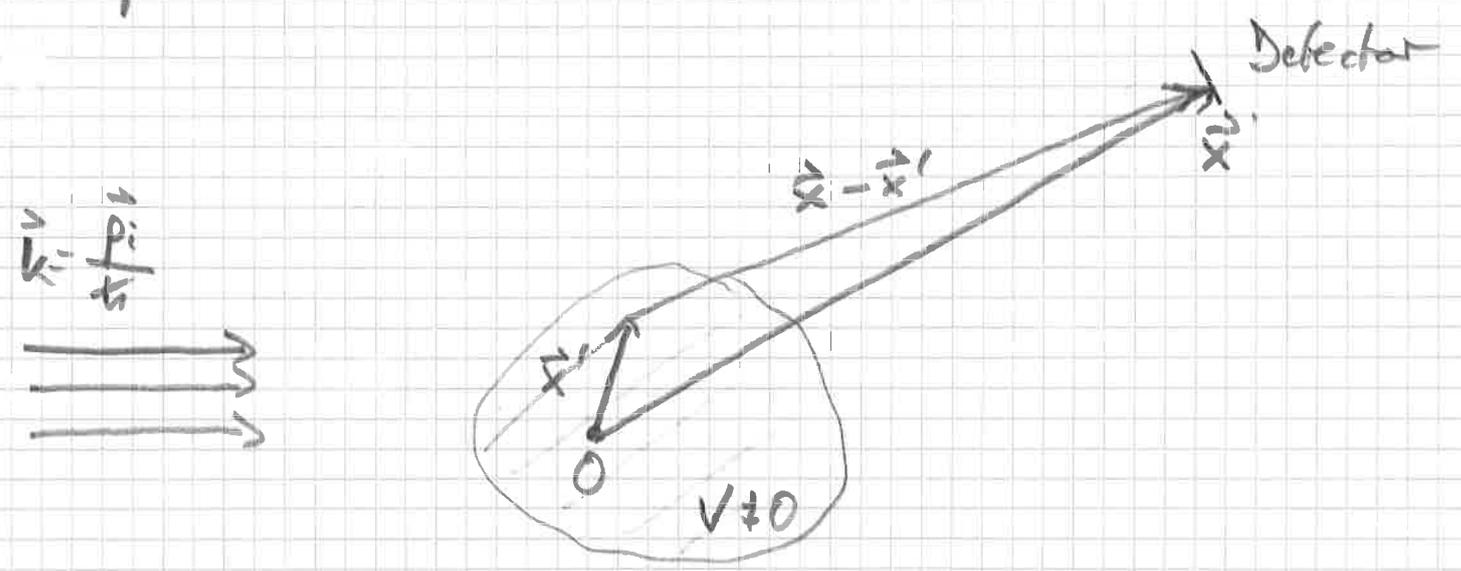
consider: only local potentials $V(\vec{x})$ (no $\frac{\partial}{\partial x}$ in V)

→ $V(\vec{x})$ diagonal in position space

$$\langle \vec{x}' | V | \vec{x}'' \rangle = V(\vec{x}') \delta^3(\vec{x}' - \vec{x}'')$$

$$\langle \vec{x} | \gamma_{\pm} \rangle = \langle \vec{x} | \phi \rangle - \frac{2m}{\hbar^2} \int d^3x' \frac{e^{\pm ik|\vec{x} - \vec{x}'|}}{4\pi|\vec{x} - \vec{x}'|} V(\vec{x}') \langle \vec{x}' | \gamma_{\pm} \rangle$$

Interpretation



Potential of finite extent, detector far away

$$|\vec{x}| \gg |\vec{x}'| \text{ with } r = |\vec{x}|, r' = |\vec{x}'|, |\vec{x} - \vec{x}'| \approx r - \hat{e}_x \cdot \vec{x}'$$

Define: $\vec{k}' = k \vec{e}_x$

$$\rightarrow e^{\pm i\vec{k}|\vec{x}-\vec{x}'|} \approx e^{\pm ikr} e^{\mp i\vec{k}'\cdot\vec{x}'}$$

$|\vec{k}\rangle$ normalized such that $\langle \vec{k} | \vec{k}' \rangle = \delta^3(\vec{k} - \vec{k}')$

$$\rightarrow \langle \vec{x} | \vec{k} \rangle = \frac{1}{(2\pi)^{3/2}} e^{i\vec{k}\cdot\vec{x}}$$

such that

$$\langle \vec{x} | \psi^\pm \rangle = \langle \vec{x} | \vec{k} \rangle - \frac{1}{4\pi} \frac{2m}{\hbar^2} \frac{e^{\pm ikr}}{r} \int d^3x' e^{\mp i\vec{k}'\cdot\vec{x}'} V(\vec{x}') \langle \vec{x}' | \psi^\pm \rangle$$

\vec{x} large

ψ^+ : outgoing spherical wave

ψ^- : incoming spherical wave

$$\langle \vec{x} | \psi^+ \rangle = \frac{1}{(2\pi)^{3/2}} \left(e^{i\vec{k}\cdot\vec{x}} + \frac{e^{ikr}}{r} f(\vec{k}, \vec{k}') \right)$$

with scattering amplitude

$$f(\vec{k}, \vec{k}') = -\frac{1}{4\pi} (2\pi)^3 \frac{2m}{\hbar^2} \int d^3x' \frac{e^{-i\vec{k}'\cdot\vec{x}'}}{(2\pi)^{3/2}} V(\vec{x}') \langle \vec{x}' | \psi^+ \rangle$$

Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{\# \text{ particles in } d\Omega \text{ per unit of time}}{\# \text{ particles incoming per unit of time and area}}$$

$$= \frac{r^2 d\Omega |\vec{q}_{out}|}{|\vec{q}_{in}|} = |f(\vec{k}, \vec{k}')|^2 d\Omega$$

Born approximation

$f(\vec{k}, \vec{k}')$ depends on $|\psi^+\rangle \rightarrow$ can only approximate

Assumption: weak interaction: $V \ll \hbar v$

$$\Rightarrow \langle \vec{x}' | \psi^+ \rangle \approx \langle \vec{x}' | \phi \rangle = \langle \vec{x}' | \vec{k} \rangle = \frac{e^{i\vec{k} \cdot \vec{x}'}}{(2\pi)^{3/2}}$$

yields: Born amplitude (1st order approximation)

$$f^{(1)}(\vec{k}, \vec{k}') = -\frac{1}{4\pi} \frac{2m}{\hbar^2} \int d^3x' e^{i(\vec{k}-\vec{k}') \cdot \vec{x}'} V(\vec{x}')$$

(Fourier transformation of $\vec{V}(\vec{x}')$ in $\vec{q} = \vec{k} - \vec{k}'$)

For rotation-symmetric potential $V(\vec{x}') = V(r')$

$$\text{use } |\vec{k} - \vec{k}'| = q = 2k \sin \frac{\theta}{2}$$

$$f^{(1)}(\theta) = -\frac{1}{2} \frac{2m}{\hbar^2} \frac{1}{iq} \int_0^\infty \frac{r^2}{r} V(r) (e^{iqr} - e^{-iqr}) dr$$

$$= -\frac{2m}{\hbar^2} \frac{1}{q} \int_0^\infty r V(r) \sin qr dr$$

- real, independent on sign of V
- for $\frac{1}{|\vec{k}|} \gg |\vec{x}'|$ (large de-Broglie wavelength)

$$f^{(1)}(\theta) = -\frac{1}{4\pi} \frac{2m}{\hbar^2} \int V(x) d^3x$$

Higher-order corrections to Born approximation

Define: Scattering operator T , such that

$$V|\psi^+\rangle = T|\phi\rangle$$

Lippmann-Schwinger eqn. reads

$$T|\phi\rangle = V|\phi\rangle + V \frac{1}{E - H_0 + i\epsilon} T|\phi\rangle \quad \text{for all } |\phi\rangle$$

→ operator equation for T

$$T = V + V \frac{1}{E - H_0 + i\epsilon} T$$

and exact scattering amplitude in T :

$$f(\vec{k}, \vec{k}') = -\frac{1}{4\pi} \frac{2\pi}{\hbar^2} (2\pi)^3 \langle \vec{k}' | T | \vec{k} \rangle$$

Iterative solution of Lippmann-Schwinger eqn.

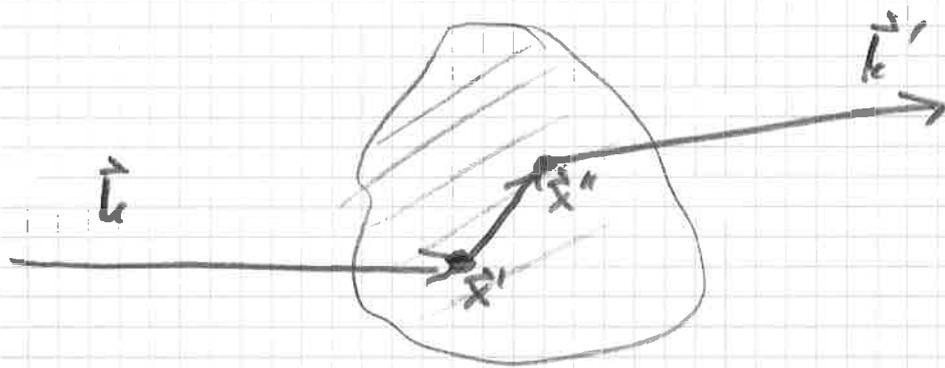
$$\begin{aligned} T &= V + V \frac{1}{E - H_0 + i\epsilon} V + V \frac{1}{E - H_0 + i\epsilon} V \frac{1}{E - H_0 + i\epsilon} V + \dots \\ &= T^{(1)} + T^{(2)} + T^{(3)} + \dots \end{aligned}$$

- 7 -

$$f(\vec{k}) = -\frac{1}{4\pi} \frac{2m}{\hbar} \int d^3x' \int d^3x'' e^{-i\vec{k}' \cdot \vec{x}'} V(\vec{x}') \cdot \left[\frac{2m}{\hbar^2} G_+(\vec{x}', \vec{x}'') \right] V(\vec{x}'') e^{i\vec{k} \cdot \vec{x}''}$$

↑ scattering in \vec{x}'

Transition $\vec{x}' \rightarrow \vec{x}''$ (propagator) ↑ scattering in \vec{x}''



Optical theorem

consider scattering amplitude at $\theta=0$ (forward)

$$f(\theta=0) = f(\vec{k}, \vec{k}) = -\frac{1}{4\pi} \frac{2m}{\hbar^2} (2\pi)^3 \langle \vec{k} | T | \vec{k} \rangle$$

compute imaginary part and use $V^\dagger = V$

$$\text{Im} \langle \vec{k} | T | \vec{k} \rangle = \text{Im} \left[\langle \vec{k} | V | \psi^+ \rangle \right]$$

$$= \text{Im} \left[\left(\langle \psi^+ | - \langle \psi^+ | V \frac{1}{E - H_0 - i\epsilon} \right) V | \psi^+ \rangle \right]$$

$$= -\pi \langle \psi^+ | V \delta(E - H_0) V | \psi^+ \rangle$$

$$= -\pi \langle \vec{k} | T^\dagger \delta(E - H_0) T | \vec{k} \rangle$$

$$= -\pi \int d^3k' \langle \vec{k}' | T | \vec{k} \rangle \langle \vec{k}' | T | \vec{k} \rangle$$

$$\delta(E - \frac{\hbar^2 k'^2}{2m})$$

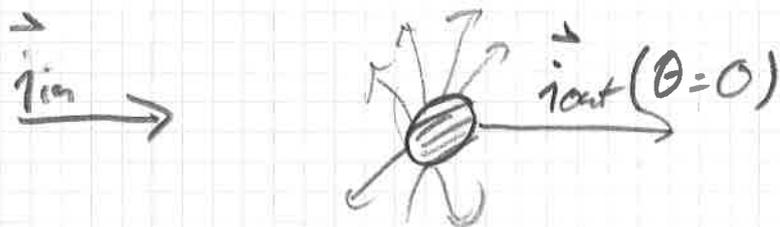
- 8 -

$$= -\pi \int d\Omega' \frac{m k}{\hbar^2} |\langle \vec{k}' | T | \vec{k} \rangle|^2$$

such that

$$\text{Im} f(\theta=0) = \frac{k \sigma_{\text{tot}}}{4\pi}$$

physical picture



absorption:

$$\frac{k_{\text{out}}}{k_{\text{in}}} \sim \text{Im}(f(\theta=0))$$

$$\text{total scattering} = \sigma_{\text{tot}}$$

Scattering attenuates forward wave

Partial wave expansion

scattering solutions for radially symmetric potential:
spherical waves.

expand plane-wave factor

$$e^{i\vec{k}\cdot\vec{x}} = \sum_{l=0}^{\infty} \sum_{m=-l}^l c_{lm}(\vec{k}) j_l(kr) Y_{lm}(\theta, \varphi)$$

for $\vec{k} \parallel \vec{e}_z$:

$$e^{ikr \cos \theta} = \sum_{l=0}^{\infty} A_l j_l(kr) Y_{l0}(\theta, \varphi)$$

$$= \sum_{l=0}^{\infty} i^l (2l+1) j_l(kr) P_l(\cos \theta)$$

expansion of scattering amplitude

$$f(\vec{k}, \vec{k}') = f(\theta) = \sum_{l=0}^{\infty} (2l+1) f_l(k) P_l(\cos \theta)$$

$$i\hbar \langle \vec{x} | \hat{\psi}^\dagger \rangle = \frac{1}{(2\pi)^{3/2}} \left(e^{i\vec{k}\cdot\vec{x}} + f(\theta) \frac{e^{ikr}}{r} \right)$$

$$= \frac{1}{(2\pi)^{3/2}} \sum_{l=0}^{\infty} (2l+1) \frac{P_l(\cos \theta)}{2il}$$

$$\left\{ \left[1 + 2ik f_l(l) \right] \frac{e^{ikr}}{r} - \frac{e^{-i(kr - l\pi)}}{r} \right\}$$

outgoing wave

incoming wave

Effect of scattering: coefficient of outgoing wave

$$1 \rightarrow 1 + 2ik f_e(k) = S_e(k) \quad \text{with } |S_e(k)|^2 = 1$$

$$= e^{2i\delta_e(k)}$$

such that

$$f(\theta) = \sum_{l=0}^{\infty} (2l+1) \frac{e^{2i\delta_l} - 1}{2ik} P_l(\cos\theta)$$

$$= \frac{1}{k} \sum_{l=0}^{\infty} (2l+1) e^{i\delta_l} \sin \delta_l P_l(\cos\theta)$$

and total cross section

$$\sigma_{tot} = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2 \delta_l$$

Interpretation: angular momentum with respect to scattering centre $L = \sqrt{l(l+1)} \hbar \approx l\hbar$



$$\hbar k d_e \approx l\hbar \Rightarrow d_e \approx \frac{l}{k}$$

for finite-range potential with range a :

• only contributions $\delta_l \neq 0$ for $l \lesssim ak$

• for $ak \ll 1$ (e.g. $k \rightarrow 0$): $\delta_0 \neq 0, \delta_{l>0} = 0$
"s-wave scattering"

Define: scattering length

$$b = - \lim_{k \rightarrow 0} f_0(k)$$

$$= - \lim_{k \rightarrow 0} \frac{\delta_0}{k}$$

yields low energy limit of scattering cross section

$$\sigma_{tot} \xrightarrow{k \rightarrow 0} \sigma_{k=0} = 4\pi b^2$$

Field theory of scattering (time-dependent processes) ⁻¹²⁻

Interaction picture

Decompose Hamiltonian operator in Schrödinger rep.

$$H_S = H_{0S} + H_{int,S}$$

Define states and operators in free Heisenberg rep
($\hat{=}$ "interaction" or "Dirac" rep)

$$\Psi_I = e^{\frac{iH_{0S}t}{\hbar}} \Psi_S$$

$$O_I = e^{\frac{iH_{0S}t}{\hbar}} O_S e^{-\frac{iH_{0S}t}{\hbar}} \quad \Rightarrow H_{0I} = H_{0S} = H_0$$

• time evolution of Ψ_I

$$i\hbar \frac{d\Psi_I}{dt} = H_{int,I} \Psi_I$$

• time evolution of O_I

$$i\hbar \frac{dO_I}{dt} = -H_0 O_I + O_I H_0 = [O_I, H_0]$$

Define: time evolution operator

$$\Psi_I(t) = U(t, t_0) \Psi_I(t_0)$$

with:

$$\bullet U(t_0, t_0) = \mathbb{1}$$

$$\bullet U(t_2, t_1) U(t_1, t_0) = U(t_2, t_0)$$

$$\bullet U^{-1}(t_0, t_1) = U(t_1, t_0)$$

$$\bullet U^\dagger(t_1, t_0) = U^{-1}(t_1, t_0) \quad \text{unitary}$$

$$\cdot i\hbar \frac{\partial}{\partial t} U(t, t_0) = H_{int, I}(t) U(t, t_0)$$

$$\Leftrightarrow U(t, t_0) = \mathbb{1} + (-i) \int_{t_0}^t dt' H_{int, I}(t') U(t', t_0)$$

solution by iteration

$$U(t, t_0) = \vec{T} \exp\left(-i \int_{t_0}^t dt' H_{int}(t')\right)$$

time-ordered product on $H_{int}(t_1) H_{int}(t_2) \dots$

Scattering matrix

S-matrix elements: probability amplitudes for transition from initial state $|i\rangle$ to final state $|f\rangle$ under interaction

with time-dependent state vector $|\Psi(t)\rangle$:

- initial state $\lim_{t \rightarrow -\infty} |\Psi(t)\rangle = |\phi_i\rangle$

$|\phi_i\rangle$: eigenstate of free hamiltonian H_0

- S-matrix element S_{fi} : projection of state vector on final state $\langle \phi_f |$

$$S_{fi} = \lim_{t \rightarrow +\infty} \langle \phi_f | \Psi(t) \rangle = \langle \phi_f | S | \phi_i \rangle$$

expressed through time-evolution operator -14-

$$S_{fi} = \lim_{t_2 \rightarrow t_0} \lim_{t_1 \rightarrow -\infty} \langle \phi_f | U(t_2, t_1) | \phi_i \rangle$$

such that: S-matrix operator

$$\begin{aligned} S &= U(\infty, -\infty) \\ &= T \exp\left(-i \int_{-\infty}^{\infty} dt' H_{int}(t')\right) \end{aligned}$$

• S unitary: $SS^\dagger = S^\dagger S = \mathbb{1}$

expressed in matrix elements:

$$\sum_i S_{fi}^* S_{fi} = \sum_f |S_{fi}|^2 = 1$$

→ conservation of norm

• sum of probabilities: $\sum_f P(i \rightarrow f) = 1$

Quantization of photon field

Maxwell equations for ED potentials (Coulomb gauge)

$$\Delta \varphi = -\frac{1}{\epsilon_0} \rho$$

$$\Delta \vec{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} = -\mu_0 \vec{j} + \frac{1}{c^2} \vec{\nabla} \left(\frac{\partial}{\partial t} \varphi \right)$$

• in absence of sources: $\rho = 0, \vec{j} = 0 \Rightarrow \varphi = 0$

$$\Delta \vec{A} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \vec{A} = 0 \quad \text{free wave equation}$$

- general solution: superposition of plane waves

$$\vec{A}(\vec{x}, t) = \sum_{\alpha=1,2} \int \frac{d^3k}{(2\pi)^3} \left(c_{\alpha}(\vec{k}, t) \vec{\epsilon}_{\alpha}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + c_{\alpha}^*(\vec{k}, t) \vec{\epsilon}_{\alpha}^*(\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right)$$

with $c_{\alpha}(\vec{k}, t) = c_{\alpha}(\vec{k}) e^{-i\omega_{\alpha} t}$

Compute Hamilton function of free field $\omega_k = c|k|$

$$H = \frac{\epsilon_0}{2} \int (c^2 \vec{B}^2 + \vec{E}^2) d^3x$$

$$= \frac{\epsilon_0}{2} \int \left[c^2 (\vec{\nabla} \times \vec{A})^2 + \left(\frac{\partial A}{\partial t} \right)^2 \right] d^3x$$

$$= 2\epsilon_0 \sum_{\alpha} \int \frac{d^3k}{(2\pi)^3} c_{\alpha}^*(\vec{k}, t) c_{\alpha}(\vec{k}, t) \omega_k^2$$

Compare to harmonic oscillator:

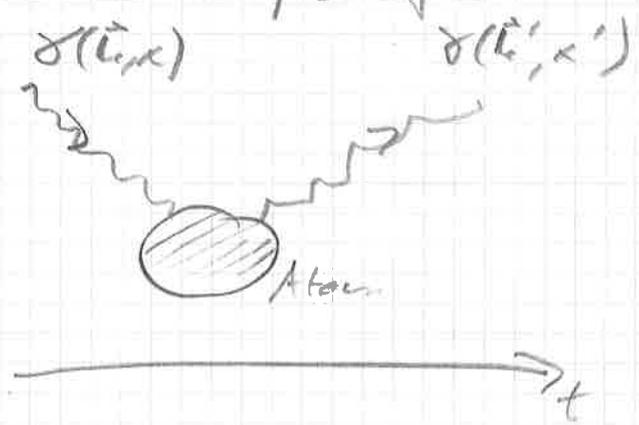
$$H = \sum_{\alpha} \int \frac{d^3k}{(2\pi)^3} \hbar \omega_k a_{\alpha}^{\dagger}(\vec{k}, t) a_{\alpha}(\vec{k}, t)$$

$$\Rightarrow a_{\alpha}(\vec{k}) e^{-i\omega_k t} = \sqrt{\frac{2\epsilon_0 \omega_k}{\hbar}} c_{\alpha}(\vec{k}) e^{-i\omega_k t}$$

Interpretation:

- Photon mode at fixed \vec{k} is described by energy quanta $\hbar\omega_k$
- $a_\alpha^\dagger(\vec{k})$ creates a photon in mode \vec{k} , polarization α ,
 $a_\alpha(\vec{k})$ annihilates : $[a, c] = [a^\dagger, c^\dagger] = 0$
 $[a, a^\dagger] = (2\pi)^3 \delta^3(\vec{k} - \vec{k}') \delta_{\alpha\alpha'}$

Scattering of light on atoms



initial state ($t = -\infty$) $|A; \vec{k}, \alpha\rangle$
 final state ($t = +\infty$) $|B; \vec{k}', \alpha'\rangle$
 are eigenstates (products) of
 $H_0 = H_0^{\text{Atom}} + H_0^{\text{field}}$

interaction Hamiltonian (electron - photon) $\Pi_i = (\vec{p} + e\vec{A})$
 Canonical momentum

$$H_{\text{int}} = \sum_{i=1}^Z \left(\frac{e}{m} \vec{A} \cdot \vec{p} + \frac{e^2}{2m} \vec{A}^2 \right)$$

Compute:

$$\langle B; \vec{k}', \alpha' | U(\infty, -\infty) | A; \vec{k}, \alpha \rangle$$

expand in powers of $\vec{A} \sim \hbar^{1/2} (a + a^\dagger)$

• $O(A)$ $|A; \vec{k}, \alpha\rangle \rightarrow \begin{matrix} |B, 0\alpha\rangle \\ |B, 2\alpha\rangle \end{matrix} \neq |B, \vec{k}', \alpha'\rangle$

• $O(A^2)$ $\frac{e^2 \vec{A}^2}{2m}$ contains aa^\dagger and $a^\dagger a$

$\int H_{\text{int}} H_{\text{int}}$ yields $\left(\frac{e}{m} \vec{A} \cdot \vec{p} \right)^2 \rightarrow aa^\dagger a^\dagger a$

Transition rate:

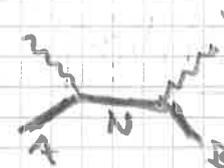
$$P = \lim_{\substack{t_i \rightarrow -\infty \\ t_f \rightarrow \infty}} \frac{1}{t_f - t_i} \left| \langle B, \vec{k}', \alpha' | U(t_f, t_i) | A, \vec{k}, \alpha \rangle \right|^2$$

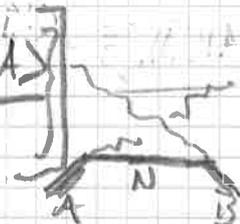
$$= \frac{1}{t_f^2} |T|^2 \frac{d^3 k'}{(2\pi)^3} \frac{1}{t_f - t_i} \left| \int_{t_i}^{t_f} dt e^{\frac{i}{\hbar} t (E_B + \hbar\omega' - E_A - \hbar\omega)} \right|^2$$

$\delta \text{im} [\vec{k}', \vec{k}' + d\vec{k}']$ Fermi's Golden rule
 $= 2\pi \delta(\omega' - \omega + \frac{E_B - E_A}{\hbar})$

with:

$$T = \frac{e^2 \hbar}{2m\epsilon_0 \sqrt{\omega\omega'}} \left[\langle B | e^{-i(\vec{k} - \vec{k}') \cdot \vec{x}} | A \rangle \vec{\epsilon}_\alpha^* (\vec{k}') \cdot \vec{\epsilon}_\alpha (\vec{k}) \right. $$

$$- \frac{1}{m} \sum_N \left\{ \frac{\langle B | e^{-i\vec{k}' \cdot \vec{x}} \vec{p} \cdot \vec{\epsilon}_\alpha^* (\vec{k}') | N \rangle \langle N | e^{i\vec{k} \cdot \vec{x}} \vec{p} \cdot \vec{\epsilon}_\alpha (\vec{k}) | A \rangle}{E_N - (E_A + \hbar\omega)} \right. $$

$$+ \frac{\langle B | e^{i\vec{k} \cdot \vec{x}} \vec{p} \cdot \vec{\epsilon}_\alpha (\vec{k}) | N \rangle \langle N | e^{-i\vec{k}' \cdot \vec{x}} \vec{p} \cdot \vec{\epsilon}_\alpha' (\vec{k}') | A \rangle}{E_N - (E_A - \hbar\omega')} \left. \right] $$

Differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{1}{c} \frac{dR}{d\Omega}$$

$$= \frac{d|\vec{k}'| |\vec{k}|^2}{(2\pi)^3} \frac{2\pi}{\hbar^2} \frac{1}{c} |T|^2 \delta(\omega' - \omega + \frac{E_B - E_A}{\hbar})$$

$$= \frac{\omega'^2}{(2\pi \hbar c)^2} |T|^2 = \left(\frac{\alpha \hbar}{mc} \right)^2 \frac{\omega'}{\omega} \left[\dots \right]^2$$

• $|\dots|^2$ is dimensionless and $O(1)$ -18-

• For $\omega_{N'} \approx \omega_N$: size of $\frac{d\sigma}{d\Omega}$ determined by

$$\tau_0^2: \tau_0 = \frac{\alpha \hbar}{mc} \approx 2.8 \cdot 10^{-15} \text{ m} \quad (\text{class. electron radius})$$

Rayleigh scattering

limiting case: • A ground state

• $\omega \ll \frac{E_N - E_A}{\hbar}$ for all $N \neq A$

$$\rightarrow B = A, \omega' = \omega$$

$$\rightarrow \frac{1}{E_N - E_A + \hbar\omega} = \frac{1}{E_N - E_A} \left(1 - \frac{\hbar\omega}{E_N - E_A} \right)$$

such that

$$|\dots| = \frac{\hbar^2 \omega^2}{m} \sum_N \frac{1}{(E_N - E_A)^3}$$

$$\cdot \langle A | \vec{p} \cdot \vec{\epsilon}_\alpha^* (\vec{k}) | N \rangle \langle N | \vec{p} \cdot \vec{\epsilon}_\alpha (\vec{k}) | A \rangle$$

$$+ \langle A | \vec{p} \cdot \vec{\epsilon} (\vec{k}) | N \rangle \langle N | \vec{p} \cdot \vec{\epsilon}_\alpha^* (\vec{k}) | A \rangle$$

$$= \frac{\omega^2}{(\omega_A^{\alpha\alpha'})^2}$$

$$\text{with } \omega^{\alpha\alpha'} \approx \frac{E_N - E_A}{\hbar} \ll \omega.$$

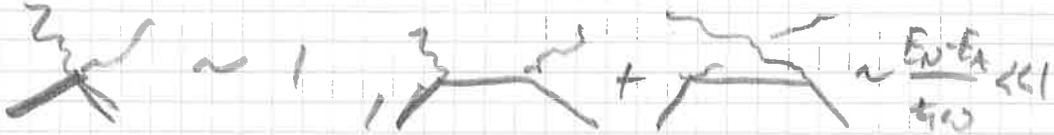
$$\rightarrow \frac{d\sigma}{d\Omega} = \tau_0^2 \left(\frac{\omega}{\omega_A^{\alpha\alpha'}} \right)^4 \quad \text{Rayleigh's law}$$

Thompson - scattering

limiting case:

$$\hbar\omega \gg E_B - E_A$$

$$\Rightarrow \omega' \approx \omega$$

Contributions: 

$$\frac{d\sigma}{d\Omega} = r_0^2 \left| \sum_{\alpha'} \vec{\epsilon}_{\alpha'}(\vec{k}') \cdot \vec{\epsilon}_{\alpha}(\vec{k}) \right|^2$$

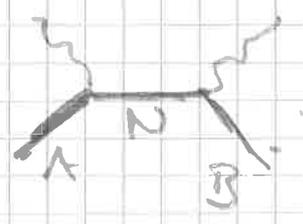
with $\frac{1}{2} \sum_{\alpha} \sum_{\alpha'}^2$:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{r_0^2}{2} (1 + \cos^2\theta) \Rightarrow \sigma = \frac{8\pi r_0^2}{3} \\ &= 6.7 \cdot 10^{-25} \text{ cm}^2 \end{aligned}$$

Resonances

assumption up to now: $E_N \neq E_A + \hbar\omega$

required to avoid divergence in



$$\sim \frac{\langle B | \dots | N \rangle \langle N | \dots | A \rangle}{E_N - (E_A + \hbar\omega)}$$

higher: time-ordered perturbation theory:

$$\int_{t_i}^{t_f} dt_2 e^{\frac{i}{\hbar} t_2 (E_N - (E_A + \hbar\omega))}$$

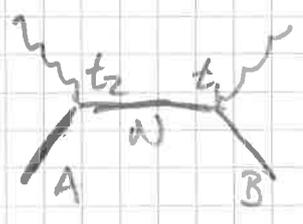
$$= \frac{\hbar}{i (E_N - (E_A + \hbar\omega))} e^{\frac{i}{\hbar} t_f (E_N - (E_A + \hbar\omega))}$$

\Rightarrow diverges for $E_N = E_A + \hbar\omega$

dominant contribution to the integral from

$$(t_1, -t_2) \lesssim \frac{\hbar}{E_N - (E_A + \hbar\omega)}$$

\Leftrightarrow typical time in state N



for $E_N \approx E_A + \hbar\omega$:

$$t_1 - t_2 \gtrsim \tau_N$$

$\tau_N = (\text{lifetime of } N)$

Impact of finite lifetime on time evolution of state N:

$$e^{-\frac{i}{\hbar} E_N t} \longrightarrow e^{-\frac{i}{\hbar} E_N t} e^{-\frac{\Gamma_N}{2\hbar} t} = e^{-\frac{i}{\hbar} (E_N - \frac{i\Gamma_N}{2}) t}$$

time evolution of stable state N

from exp decay of N:

$$|T|^2 \sim e^{-\frac{\Gamma_N}{\hbar} t}$$

lifetime $\tau_N = \frac{\hbar}{\Gamma_N}$

• Γ_N can be computed from probability for spontaneous emission $N \rightarrow B + \gamma'$

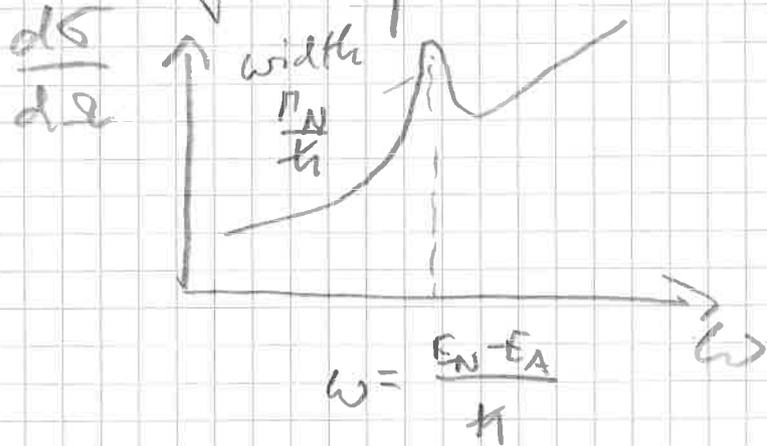
• matrix element for $\hbar\omega \approx E_N - E_A$

$$|T|^2 = \left(\frac{e^{2t/\hbar}}{2\pi\epsilon_0 \sqrt{\hbar\omega}} \right)^2 \frac{1}{\omega^2} \left| \frac{\langle B | e^{-i\hat{a} \cdot \vec{x}} \hat{p} \cdot \vec{e}_\alpha (k') | N \rangle \langle N | e^{i\hat{a} \cdot \vec{x}} \hat{p} \cdot \vec{e}_\alpha (k) | A \rangle}{E_N - (E_A + \hbar\omega) - i \frac{\Gamma_N}{2}} \right|^2$$

scattering cross section:

$$\frac{d\sigma}{d\Omega} = \tau_0^2 \frac{\omega'}{\omega} \frac{1}{\omega^2} \frac{|\langle B | \dots | N \rangle|^2 |\langle N | \dots | A \rangle|^2}{(E_N - (E_A + \hbar\omega))^2 + \frac{\Gamma_0^2}{4}}$$

yields Lorentz-curve superimposed on regular energy dependence



for $\omega \neq \frac{E_N - E_A}{\hbar}$: neglect Γ_N

typically (atomic physics): $\Gamma_N \approx 10^{-7} (E_N - E_{N'})$

$$(\Gamma \sim \alpha^5 m_e, \quad E = R_H \approx \alpha^2 m_e)$$

physical picture:

absorption of photon, followed by spontaneous emission

$$\frac{dG}{d\Omega} = \frac{\text{absorption probability}}{\text{incoming flux density}} \cdot \frac{\text{emission probability}}{\text{solid angle}}$$

→ resonance fluorescence