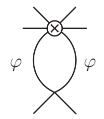
## 1.4 Matching

In the previous section we established that in order to construct an EFT, we split the degrees of freedom of the UV theory into hard and soft modes and remove hard modes, which we defined in connection with a cutoff $\Lambda$. The difference between the effective and the UV theory were the absence of the hard modes and the presence of new interactions of the soft modes in the EFT. These new interactions came with to-be-determined coupling constants, the Wilson coefficients. In a top-down EFT, i.e. when the UV theory is known, these constants can be determined from the following line of reasoning: The UV theory and the EFT must agree in the IR, but will generally differ in the UV.

The fact that they differ is almost too trivial to dwell upon: Processes in the deep UV can involve external fields that we classified as hard modes before, which can obviously not be generated in the EFT. But even scattering processes between the soft modes at hard-scale momentum transfers will in general produce different results. This is due to the fact that the effective interactions in the EFT are constructed in a local OPE, since we argued that the soft modes cannot probe the small non-locality in the short-distance effects generated by hard physics. In the language of Feynman diagrams, we expanded the propagator as

$$\frac{1}{k^2 - M^2} = -\frac{1}{M^2}\left[1 + \mathcal{O}\left(\frac{k^2}{M^2}\right)\right] \tag{1.1}$$

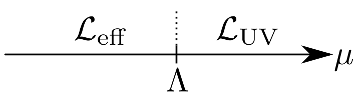
At high energies, this expansion is no longer valid and the effective description breaks down. This breakdown is particularly obvious when computing matrix elements at loop-order, where the EFT will produce matrix elements that are more divergent in the UV than their corresponding counterparts from the UV theory. This can be seen at the example in eq. (1.6). In the effective theory, this graph becomes:




$$= \frac{\lambda\,\mathcal{C}_6}{\Lambda^2}\int \frac{d^dl}{(2\pi)^d}\left(\frac{1}{l^2 - m^2}\right)^2 = \frac{i\lambda\,\mathcal{C}_6}{16\pi^2\Lambda^2}\left\{\frac{1}{\epsilon} + \log\frac{\mu^2}{m^2}\right\}\,, \tag{1.2}$$

where we worked in dimensional regularization with $d = 4 - 2\epsilon$. The quantity $\mathcal{C}_6$ is the Wilson coefficient of the effective $\phi^6$ interaction. The ultraviolet divergence present in this graph was absent in eq. (1.6). It originates from regions in the integration over $l$ where the virtual modes becomes hard enough to probe the non-locality in the effective vertex. In this region, the EFT breaks down and produces UV divergences. We can actually make use of that fact later to solve the problems of the aforementioned large logarithms.

We have yet to determine the Wilson coefficients of the EFT. To this end, let us return to the statement from which we started and focus on the part of it we have not discussed: The UV theory and the EFT must agree in the IR.



At the cutoff scale $\Lambda$ at which we integrated out the hard modes, $\mathcal{L}_{\text{UV}}$ and $\mathcal{L}_{\text{eff}}$ must produce consistent matrix elements. Given that the full theory $\mathcal{L}_{\text{UV}}$ is known, this fixed the couplings of $\mathcal{L}_{\text{eff}}$ order by order in power-counting and in perturbation theory. This producedure is called *matching* and the cutoff scale, $\Lambda$ is often also called *matching scale*.

Matching can be done through several methods. By far the most common one is diagrammatic. One computes matrix elements in both the effective and UV theories and equates them to detmermine the coupling constants of the effective Lagrangian. Note that this does not only apply to the Wilson coefficients of the "new" effective operators but also to the coupling constants of operators that both $\mathcal{L}_{\text{eff}}$ and $\mathcal{L}_{\text{UV}}$ share. For example, the coupling $\lambda$ in our Lagrangian (1.4) will not be the same as the $\lambda$ in the UV Lagrangian (1.2). Instead, it will receive corrections from virtual hard modes.

Another way of performing the matching is the background field method. In this method, fields are separated into the classical fields and quantum fluctuations. One can then integrate out the hard modes by solving the path integral for them explicitely. See section 1.6 for an introduction.

### 1.4.1 Matching at tree-level: Muon decay

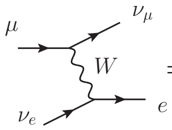
A classic example of an effective theory is the Fermi theory of muon decay. The decay of the muon $\mu \to e\bar{\nu}_e\nu_\mu$ proceeds through a virtual $W$ boson in the SM. The momentum transfer is however much lower than the mass of the $W$ boson, $m_W = 80$ GeV, since it is governed by the muon mass $m_\mu = 105$ MeV. Therefore, the interaction is well described by an effective four-fermion interaction of the form:

$$\mathcal{L}_{\text{eff}} \supset -\frac{4G_F}{\sqrt{2}}(\bar{\nu}_\mu\gamma^\alpha P_L\mu)(\bar{e}\gamma_\alpha P_L\nu_e)\,, \tag{1.3}$$

where $G_F$ is Fermi's constant. By power-counting the fields we can see that it has mass dimension $G_F \sim \mathcal{O}(\text{GeV}^{-2})$. To connect it to our previous (and nowadays more standard) notation it corresponds to:

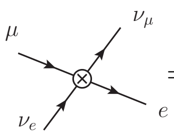
$$-\frac{4G_F}{\sqrt{2}} \equiv \frac{\mathcal{C}_{\mu e}}{\Lambda^2}\,. \tag{1.4}$$

Before the $W$ boson was discovered, $G_F$ would be a constant extracted from measuring the muon decay rate. Today, we know the UV theory to the Lagrangian (1.3) - the Standard Model. We can thus *match* the SM to the Fermi theory and compute $G_F$ in terms of parameters of the SM. To this end, let us compute the matrix elements of muon decay in the full theory and the effective theory. In the SM we have:




$$
\begin{aligned}
&= \bar{u}(p_\nu)\left(\frac{ig}{\sqrt{2}}\gamma^\alpha P_L\right)u(p_\mu)\cdot\frac{-ig_{\alpha\beta}}{(k_e+k_\nu)^2-m_W^2}\cdot\bar{u}(k_e)\left(\frac{ig}{\sqrt{2}}\gamma_\alpha P_L\right)v(k_\nu) \\
&= \frac{ig^2}{2(q^2-m_W^2)}\left[\bar{u}(p_\nu)\gamma^\alpha P_L u(p_\mu)\right]\left[\bar{u}(k_e)\gamma_\alpha P_L v(k_\nu)\right] \\
&= -\frac{ig^2}{2m_W^2}\left[\bar{u}(p_\nu)\gamma^\alpha P_L u(p_\mu)\right]\left[\bar{u}(k_e)\gamma_\alpha P_L v(k_\nu)\right]+\mathcal{O}\left(\frac{q^2}{m_W^2}\right).
\end{aligned}
\tag{1.5}
$$

In the second line we have defined $q = k_e + k_\nu$ and in the last line we have expanded around $m_W \gg q^\mu$. The next step is to compute the matrix element for this process in the effective theory. It can be directly read off from the Lagrangian (1.3):




$$
= -\frac{4iG_F}{\sqrt{2}}\left[\bar{u}(p_\nu)\gamma^\alpha P_L u(p_\mu)\right]\left[\bar{u}(k_e)\gamma_\alpha P_L v(k_\nu)\right]. \tag{1.6}
$$

Equating (1.5) and (1.6), we find:

$$
G_F = \frac{g^2}{4\sqrt{2}m_W^2}\,, \quad \text{or alternatively} \quad \mathcal{C}_{\mu e} = -\frac{g^2}{2}\,. \tag{1.7}
$$

Here we set $\Lambda = m_W$ is the cutoff scale of the Fermi theory: It looses validity at momentum transfers close to the $W$-boson mass. We have computed the leading-power matching of the muon decay to the Fermi theory.

### 1.4.2 Matching at one-loop and the method of regions

Let us now carry out an example at loop-level. As a toy theory, we will go back to our model of two real scalars. However, we will modify the way in which they interact:

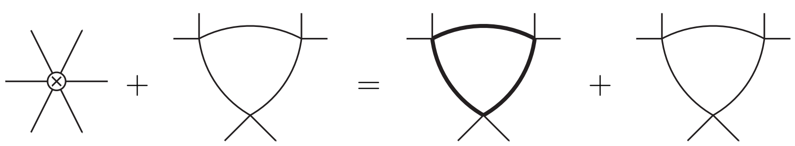
$$
\mathcal{L}_{\text{int}} = \frac{\lambda_1}{4!}\varphi^4 + \frac{\lambda_2}{4}\varphi^2\Phi^2\,. \tag{1.8}
$$

Now after removing the heavy scalar $\Phi$, the effective Lagrangian has an operator to generate the six-point function for $\varphi$:

$$
\mathcal{L}_{\text{eff}} \supset \frac{\mathcal{C}_6}{6!}\varphi^6\,. \tag{1.9}
$$

The one-loop matching to this operator gives us the following diagrammatic matching equation:

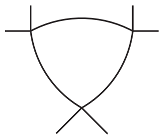
$$
i\mathcal{A}_{EFT} \qquad = \qquad i\mathcal{A}_{\text{full}}
$$




(1.10)

Before doing even a single line of computation, we seem to see that the second diagram on each side of the equation will give an identical contribution and will cancel out of the matching. The Wilson coefficient $\mathcal{C}_6$ would therefore be purely determined from the loop diagram with a heavy scalar. This seems reasonable, since we have stated earlier that the Wilson coefficients encode the hard-scale physics that we have integrated out from the theory. On the other hand, our previous comment about the origin of UV divergences in the EFT matrix elements should remind you of the fact that there are also regions of the loop integration in which $l^2 \sim \Lambda^2$ and thus the light fields become highly virtual - i.e. hard - modes. To see that these do not give matching corrections, we need to look at the loop integral a little more carefully.

The contribution from light fields in the full theory yields an amplitude of the following form:



$$= -\lambda_2^2 N \int \frac{d^d l}{(2\pi)^d} \left( \frac{1}{l^2 - m^2} \right)^3 , \tag{1.11}$$

where $N$ is a symmetry factor from interchanging the extern legs. To continue, we split the loop integral into the regions where the virtual fields are hard and soft:

$$\begin{aligned} I_{\text{full}} &= I_H + I_S \\ \int \frac{d^d l}{(2\pi)^d} \left( \frac{1}{l^2 - m^2} \right)^3 &= \left[ \int \frac{d^d l}{(2\pi)^d} \left( \frac{1}{l^2 - m^2} \right)^3 \right]_H + \left[ \int_S \frac{d^d l}{(2\pi)^d} \left( \frac{1}{l^2 - m^2} \right)^3 \right]_S . \end{aligned} \tag{1.12}$$

In the effective theory, this diagram looks identical, but should not have the hard modes propagating in the loop, so it is given purely by $I_S$. The full theory on the other hand has both $I_S$ and $I_H$. Consequently, the contribution from $I_S$ cancels in the matching and the contribution to the Wilson coefficient is given by the piece $I_H$ only. This, again, is intuitive: The Wilson coefficients are supposed to encode the hard dynamics only. Soft physics are described by the dynamic degrees of freedom in the effective theory. We have already arrived at an important conclusion:

The Wilson coefficients are fully determined by the hard region of the full theory amplitude.

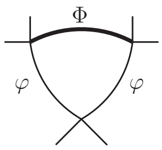
The last thing we need to work out is the computation of $I_H$. In dimensional regularization, the computation is done by simply power-counting $l^2$ of the hard scale, meaning $l^2 \sim \Lambda^2 \gg m^2$ and expanding the integrand around this limit. At leading power we obtain:

$$\left[ \int \frac{d^d l}{(2\pi)^d} \left( \frac{1}{l^2 - m^2} \right)^3 \right]_H = \int \frac{d^d l}{(2\pi)^d} \left( \frac{1}{l^2} \right)^3 + \mathcal{O}\left( \frac{m^2}{\Lambda^2} \right) . \tag{1.13}$$

This integral is completely scaleless and thus vanishes in dimensional regularization. There is no matching correction from the hard region of the light fields.

### 1.4.3 Closing remarks on the method of regions

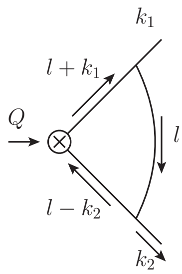
Before we move on to renormalization, we should go through a list of important remarks. If we look at tree-level matching from a diagrammatic perspective, it might seem that the corresponding diagrams in the effective theory are the same as the ones in the full theory, but with the hard propagators contracted to a point. This is of course not true: Loop diagrams with both hard and soft modes propagating in the loop have hard regions. For example, the loop diagram we saw earlier in the hard region is:




$$= -\lambda g^2 N \left[\int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 - M^2} \left(\frac{1}{l^2 - m^2}\right)^2\right]_H \qquad (1.14)$$
$$= -\lambda g^2 N \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2 - M^2} \left(\frac{1}{l^2}\right)^2 .$$

This integral is not scaleless and thus gives a contribution to the Wilson coefficient of the effective six-point operator.

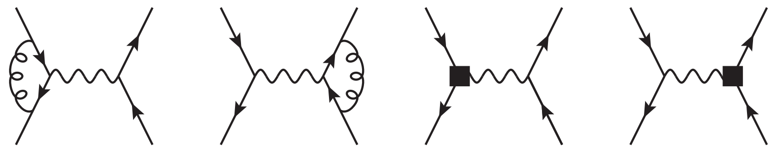
Another remark is that there are can in general be more than just hard and soft regions. For example, for processes at high energies, where the external momenta have large components, there are regions in which only individual light fields become hard. Consider a process of a heavy particle decaying to two very light particles, with momenta $k_1^2 \sim k_2^2 \sim 0$, but $(k_1 + k_2)^2 = Q^2$. Then at one-loop we have an integral of the form:




$$\sim \int \frac{d^d l}{(2\pi)^d} \frac{1}{l^2} \frac{1}{(l + k_1)^2} \frac{1}{(l - k_2)^2} . \qquad (1.15)$$

This integral has regions in which $l^\mu \propto k_i^\mu$, and consequently $l^2 \sim 0$ without every component of $l^\mu$ being small. At the example of $l^\mu \propto k_1^\mu$, this means that the propagators with $l^2$ and $(l + k_1)^2$ count as soft while $(l - k_2)^2 \sim (k_1 - k_2)^2 \sim Q^2$ count as hard. These regions are called *collinear regions* and are relevant regions for SCET, an effective theory we mentioned a few times before.

Finally, we should address a subtlety about the nature of the divergences we encounter in these computations. We have established that the matching coefficients are determined by the hard region. In our computation, we have not distinguished between IR and UV divergences. Consider the case of the weak decay of a quark, mediated through a $W$ boson, similar to the muon decay discussed above. The difference is that a four-quark transition receives important corrections from QCD. When computing these diagrams in the full theory, some diagrams produce UV divergences that are cancelled by the counterterms. In the example at hand, the diagrams in question are:



Defining the tree-level amplitude as $i\mathcal{A}_0$, each counterterm graph gives

$$i\mathcal{A}_{\delta_g} = i\mathcal{A}_0 \left( -\frac{\alpha_s C_F}{4\pi} \frac{1}{\epsilon_{\rm UV}} \right) , \tag{1.16}$$

irrespective of the region we are considering. The corresponding diagram producing the divergence that this term is supposed to cancel does not exist in the matching computation - it is scaleless and thus vanishes. We seem to have a leftover UV divergence. The resolution to this puzzle is that we were not careful enough about the scaleless integrals. They vanish in dimensional regularization only if we do not distinguish between IR and UV divergences. Treating them more carefully, each of the two vertex-correction diagrams gives a contribution

$$i\mathcal{A}_V = i\mathcal{A}_0 \left( \frac{\alpha_s C_F}{4\pi} \left[ \frac{1}{\epsilon_{\rm UV}} - \frac{1}{\epsilon_{\rm IR}} \right] + \mathcal{O}(\epsilon^0) \right) . \tag{1.17}$$

Thus, these graphs cancel the UV divergences from the counterterms, leaving an IR divergence. Equally, there will be matrix elements in the effective theory yielding scaleless integrals, whose UV divergences will be cancelled by counterterms for the effective operators. In the end, the matching coefficients will only contain IR divergences, as they should - see the next section.