MMP I
Solution Sheet 1

HS 21
Prof. Ph. Jetzer

L. Buonocore, M. Loechner, X. Liu, M. Ebersold
https://www.physik.uzh.ch/en/teaching/PHY312

Issued: 23.09.2021
Due: 30.09.2021

Exercise 1 [Fouries series (5 points)]

a)

$$f(x) = (x^2 - \pi^2)^2, \quad |x| \leqslant \pi$$
$$f(x) = f(-x) \Rightarrow \text{even function!} \Rightarrow b_n = 0 \quad \forall n$$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi}(x^2-\pi^2)^2 dx \qquad \text{because } f \text{ is even} \\ &= \frac{2}{\pi}\int_0^{\pi}(x^4 - 2x^2\pi^2 + \pi^4)dx = \frac{16}{15}\pi^4 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi}(x^4 - 2x^2\pi^2 + \pi^4)\cos(nx)dx \\ &\text{(partial integration)} \\ &= \frac{2}{\pi}\left[(x^4 - 2x^2\pi^2 + \pi^4)\frac{1}{n}\sin(nx)\right]_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi}(4x^3 - 4x\pi^2)\sin(nx)dx \\ &= \frac{8}{n^2\pi}\left[(x^3 - x\pi^2)\cos(nx)\right]_0^{\pi} - \frac{8}{n^2\pi}\int_0^{\pi}(3x^2 - \pi^2)\cos(nx)dx \\ &= -\frac{8}{n^3\pi}\left[(3x^2 - \pi^2)\sin(nx)\right]_0^{\pi} + \frac{8}{n^3\pi}\int_0^{\pi} 6x\sin(nx)dx \\ &= -\frac{48}{n^2\pi}\left[x\cos(nx)\right]_0^{\pi} + \frac{48}{n^4\pi}\int_0^{\pi}\cos(nx)dx \\ &= -\frac{48}{n^4\pi}\pi\cos(n\pi) = \frac{48}{n^4}(-1)^{n+1} \end{aligned}$$

$$\Rightarrow f(x) = (x^2 - \pi^2)^2 = \frac{8}{15}\pi^4 + \sum_{n=1}^{\infty}\frac{48}{n^4}(-1)^{n+1}\cos(nx)$$

We want to isolate the $\sum \frac{1}{n^4}$ term. To do that, let us have a look at $x = \pi$:

$$\text{Normal function:} \quad f(\pi) = (\pi^2 - \pi^2) = 0$$

$$\begin{aligned}
\text{Fourier series:} \quad f(\pi) &= \frac{8}{15}\pi^4 + \sum_{n=1}^{\infty} \frac{48}{n^4}(-1)^{n+1}\cos(n\pi) \qquad \text{with } \cos(n\pi) = (-1)^n \\
&= \frac{8}{15}\pi^4 + \sum_{n=1}^{\infty} \frac{48}{n^4}(-1)^{2n+1} \\
&= \frac{8}{15}\pi^4 - 48\sum_{n=1}^{\infty} \frac{1}{n^4} \\
\Rightarrow 0 &= \frac{8}{15}\pi^4 - 48\sum_{n=1}^{\infty} \frac{1}{n^4} \\
\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} &= \frac{\pi^4}{90}
\end{aligned}$$

b)

$$h(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \\ \frac{1}{2}, & x = 0 \text{ or } x = \pi \end{cases}, \qquad 2\pi\text{-periodic}$$

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} h(x)dx = \frac{1}{\pi}\int_0^{\pi} 1dx = 1 \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} h(x)\cos(nx)dx = \frac{1}{\pi}\int_0^{\pi} \cos(nx)dx = \frac{1}{n\pi}\left[\sin(nx)\right]_0^{\pi} = 0 \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} h(x)\sin(nx)dx = \frac{1}{\pi}\int_0^{\pi} \sin(nx)dx = -\frac{1}{n\pi}\left[\cos(nx)\right]_0^{\pi} \\
&= -\frac{1}{n\pi}[\cos(n\pi) - 1] = \frac{1}{n\pi}\left[(-1)^{n+1} + 1\right] \\
\Rightarrow h(x) &= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n\pi}\left[(-1)^{n+1} + 1\right]\sin(nx)
\end{aligned}$$

$$h(0) = \frac{1}{2}, \quad \text{even if we define } h(x) \text{ as } \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$$

Exercise 2 [Series (2 points)]

a)

$$f(x) = x \qquad \text{in } -\pi < x < \pi$$

$$\text{odd function } (f(x) = -f(-x)) \Rightarrow a_n = a_0 = 0$$

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx = -\frac{1}{n\pi}\left[x\cos(nx)\right]_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos(nx)dx \\ &= -\frac{1}{n\pi}\left[\pi\cos(n\pi) + \pi\cos(-n\pi)\right] + \frac{1}{n^2\pi}\left[\sin(nx)\right]_{-\pi}^{\pi} \\ &= -\frac{2}{n}\cos(n\pi) = -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1} \\ \Rightarrow x &= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx) \end{aligned}$$

b) Evaluate at $x = \frac{\pi}{2}$:

$$\begin{aligned} \frac{\pi}{2} =& 2\Big[\frac{(-1)^2}{1}\sin(\frac{\pi}{2}) + \frac{(-1)^3}{2}\sin(\pi) + \frac{(-1)^4}{3}\sin(\frac{3\pi}{2}) + \\ &+ \frac{(-1)^5}{4}\sin(2\pi) + \frac{(-1)^6}{5}\sin(\frac{5\pi}{2}) + ...\Big] \\ =& 2[1 - \frac{1}{3} + \frac{1}{5} + ...] \\ \Rightarrow \frac{\pi}{4} =& 1 - \frac{1}{3} + \frac{1}{5} + ... \end{aligned}$$

Exercise 3 [Trigonometric series (4 points)]

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \text{ (sum function)}$$

$$s_n(x) = \sum_{k=-n}^{n} c_k e^{ikx} \text{ (partial sum)}$$

- **properties**:

  (i) $\forall\delta > 0 : s_n(x)$ converge uniformly in $x \in [-\pi + \delta, \pi - \delta]$

  (ii) $\exists k : |s_n(x)| \leqslant k \quad \forall x \forall n$

- **to show:**
  $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ is the Fourier series of its sum function $f(x)$ in $x \in [-\pi, \pi]$

fix some n and consider:

coefficients of the sum function: $c_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} s_n(x)e^{-imx}dx$
$\forall m \leqslant n$ (such that c_m appears in s_n)

coefficients of the Fourier series: $a_m = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-imx}dx$

to show: $a_m = c_m \Longleftrightarrow a_m - c_m = 0$

look at

$$|a_m - c_m| = \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(f(x) - s_n(x)\right) e^{-imx}dx\right|$$
$$\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x) - s_n(x)| \left|e^{-imx}\right| dx$$
$$\text{use: } \left|e^{-imx}\right| = 1$$
$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-\pi+\delta} |f(x) - s_n(x)| \, dx + \int_{-\pi+\delta}^{\pi-\delta} |f(x) - s_n(x)| \, dx + \int_{\pi-\delta}^{\pi} |f(x) - s_n(x)| \, dx\right]$$

integral $\displaystyle\int_{-\pi+\delta}^{\pi-\delta} |f(x) - s_n(x)| \, dx$:

$\forall \delta > 0 \exists n \geqslant m : |f(x) - s_n(x)| < \delta$ for $x \in [-\pi + \delta, \pi - \delta]$ (property 1)

$$\Rightarrow \int_{-\pi+\delta}^{\pi-\delta} |f(x) - s_n(x)| \, dx < \delta \int_{-\pi+\delta}^{\pi-\delta} dx = \delta[2\pi - 2\delta] = 2\pi\delta + O(\delta^2)$$

integral $\displaystyle\int_{\pi-\delta}^{\pi} |f(x) - s_n(x)| \, dx$:

$\exists k$ with $|s_n(x)| < k \ \forall n \ \forall x$ (property 2)

$\Rightarrow |f(x) - s_n(x)| \leqslant |f(x)| + |s_n(x)| < 2k$ since $|f(x)| < k$ and $|s_n(x)| < k$

$$\Rightarrow \int_{\pi-\delta}^{\pi} |f(x) - s_n(x)| \, dx < 2k \int_{\pi-\delta}^{\pi} dx = 2k\delta = \int_{-\pi}^{\pi+\delta} |f(x) - s_n(x)| \, dx$$

$$\Rightarrow |a_m - c_m| < \frac{1}{2\pi}[4k\delta + 2\pi\delta] + O(\delta^2)$$
$$= \delta[\frac{2k}{\pi} + 1] + O(\delta^2) \qquad \forall \delta > 0$$

$$\rightarrow \text{make } \delta \text{ arbitrary small}$$
$$\Rightarrow |a_m - c_m| \leqslant 0 \Rightarrow |a_m - c_m| = 0 \qquad \square$$

Exercise 4 [Convergence (2 points)]

a) It is clear that, for $|z| \geqslant 1$, the series does not converge.
Consider now the partial sums $S_k := 1 + z + ... + z^k$
We have that

$$S_k = 1 + z(1 + ... + z^{k-1}) = 1 + z(S_k - z^k)$$
$$\Rightarrow S_k = \frac{1 - z^{k+1}}{1 - z}$$

For $|z| < 1$, the partial sums converge pointwise towards f:

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1 - z^{k+1}}{1 - z} = \frac{1}{1 - z} = f(z)$$

b) First, we show that f does not converge uniformly in $\{z \in \mathbb{C}, |z| < 1\}$.
Fix a $0 < \epsilon < \frac{1}{2}$ and a $k \in \mathbb{N}$. Consider z with $1 > |z| \geqslant 1 - \delta$. Then

$$|f - S_k| = \left|\frac{z^{k+1}}{1 - z}\right| \geqslant \frac{(1 - \delta)^{k+1}}{2}$$

This can be made larger than ϵ, if you choose δ sufficiently small (and thus $|z|$ sufficiently close to 1).
Now we fix a $0 < \delta < 1$ and show, in a similar way that the series converges uniformly in $\{z \in \mathbb{C}, |z| < 1 - \delta\}$.

$$|f - S_k| = \left|\frac{z^{k+1}}{1 - z}\right| \leqslant \frac{(1 - \delta)^{k+1}}{\delta}$$

Choosing a sufficiently large k, we can make this expression to become smaller than any $\epsilon > 0$.