Exercise 1. ***Born approximation and low-energy soft-sphere scattering***

a) In the Born approximation the scattering amplitude reads:

$$f^{(1)}(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int \mathrm{e}^{i(\vec{k}'-\vec{k})\cdot\vec{r}_0} V(\vec{r}_0)\, d^3\vec{r}_0 \tag{1}$$

For low-energy (long wave-length) scattering the exponential factor is essentially constant in the scattering region and the Born approximation simplifies to:

$$f^{(1)}(\theta, \phi) \approx -\frac{m}{2\pi\hbar^2} \int V(\vec{r}_0)\, d^3\vec{r}_0. \tag{2}$$

Consider the scattering off a soft-sphere potential

$$V(\vec{r}) = \begin{cases} V_0, & \text{if } r \leq a \\ 0, & \text{if } r > a \end{cases}$$

i) Compute the low-energy scattering amplitude.

ii) Compute the differential cross section.

iii) Compute the total cross section.

b) In the lecture you have seen how to derive the Born approximation starting from the integral representation of the Schroedinger equation.

Extending that reasoning derive a formal expression for the second-order Born approximation.

c) Consider again the scattering off a soft-sphere potential as in part a) of the exercise. Compute the scattering amplitude in the second Born approximation.

Exercise 2. ***One-dimensional Green's function***

Consider the one-dimensional Schroedinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x) \tag{3}$$

Using the Green's function method, show that the integral form of the solution is

$$\psi(x) = \psi_0(x) - \frac{im}{\hbar^2 k}\int_{-\infty}^{\infty} e^{ik|x-y|} V(y)\psi(y) dy \tag{4}$$

where ψ_0 is the solution of the free Schroedinger equation, and $k = \sqrt{2mE/\hbar^2}$.