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Applications of General Relativity

Sheet 9

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Website: `http://www.physik.uzh.ch/en/teaching/PHY519/`

Exercise 1 [Post-Newtonian Lagrangian for a binary system]
The gravitational action for a particle a propagating in the metric $g_{\mu\nu}$ is given by

$$S_a = -m_a c \int \mathrm{d}t \left(-g_{\mu\nu} \frac{\mathrm{d}x_a^\mu}{\mathrm{d}t} \frac{\mathrm{d}x_a^\nu}{\mathrm{d}t} \right)^{1/2} . \tag{1}$$

In this problem we are considering two masses m_1 and m_2 with separation r. Note that the total action is given by $S = S_1 + S_2$ and that particle 1 propagates in the metric generated by particle 2 and vice versa.

(i) Expand the action to order 1PN.

(ii) Expand the energy momentum tensor

$$T_a^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\mathrm{d}t}{\mathrm{d}\tau_a} m_a \frac{\mathrm{d}x_a^\mu}{\mathrm{d}t} \frac{\mathrm{d}x_a^\nu}{\mathrm{d}t} \delta^{(3)} \left(\boldsymbol{x} - \boldsymbol{x}_a(t) \right) \tag{2}$$

to order 1PN. Here τ_a is the proper time of particle a.

(iii) Calculate the required metric corrections from the above energy momentum tensor. Recall that the metric is given order by order as

$$^{(2)}g_{00} = -2\phi \qquad\qquad ^{(2)}g_{ij} = -2\phi\delta_{ij} \tag{3}$$

$$^{(3)}g_{0i} = \zeta_i + \partial_i \partial_0 \chi \qquad\qquad ^{(4)}g_{00} = -2\left(\phi^2 + \psi\right) \tag{4}$$

where ϕ, ζ_i and ψ are defined as

$$\phi(t, \boldsymbol{x}) = -\frac{G}{c^4} \int \frac{\mathrm{d}^3 x'}{|\boldsymbol{x} - \boldsymbol{x}'|} \; ^{(0)}T^{00}(t, \boldsymbol{x}') \tag{5}$$

$$\zeta_i(t, \boldsymbol{x}) = -\frac{4G}{c^4} \int \frac{\mathrm{d}^3 x'}{|\boldsymbol{x} - \boldsymbol{x}'|} \; ^{(1)}T^{0i}(t, \boldsymbol{x}') \tag{6}$$

$$\psi(t, \boldsymbol{x}) = -\frac{G}{c^4} \int \frac{\mathrm{d}^3 x'}{|\boldsymbol{x} - \boldsymbol{x}'|} \left[^{(2)}T^{00}(t, \boldsymbol{x}') + ^{(2)} T^{ii}(t, \boldsymbol{x}') \right] \tag{7}$$

$$\chi(t, \boldsymbol{x}) = -\frac{G}{2c^4} \int \mathrm{d}^3 x' \left| \boldsymbol{x} - \boldsymbol{x}' \right| \; ^{(0)}T^{00}(t, \boldsymbol{x}') \tag{8}$$

Note that these definitions follow the book of Straumann whereas the script follows Weinberg/Maggiore.

(iv) Plug the metric perturbations into the action and identify the Newtonian and leading order post-Newtonian Lagrangian.