Exercise 1. ***Electron in a time-dependent magnetic field***

Imagine an electron at rest at the origin, in the presence of a magnetic field whose magnitude is constant, but whose direction sweeps out a cone, of opening angle α, at a constant angular velocity ω:

$$\vec{B}(t) = B_0 \left[\sin\alpha \cos\omega t\, \hat{\imath} + \sin\alpha \sin\omega t\, \hat{\jmath} + \cos\alpha\, \hat{k} \right]. \tag{1}$$

The Hamiltonian is

$$H(t) = \frac{e}{m}\vec{B}\cdot\vec{S} = \frac{\hbar\omega_1}{2}\begin{pmatrix} \cos\alpha & \mathrm{e}^{-i\omega t}\sin\alpha \\ \mathrm{e}^{i\omega t}\sin\alpha & -\cos\alpha \end{pmatrix}, \tag{2}$$

where $\omega_1 = \frac{eB_0}{m}$.

(a) Compute the normalized eigenspinors of $H(t)$:

$$H(t)\chi_\pm(t) = E_\pm\chi_\pm,$$

and show that

$$E_\pm = \pm\frac{\hbar\omega_1}{2}.$$

(b) Supposing that the electron starts out with spin up along $\vec{B}(0)$:

$$\chi(0) = \begin{pmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{pmatrix},$$

verify that the exact solution of the time-dependent Schroedinger equation can be written as:

$$\chi(t) = \left[\cos\left(\frac{\lambda t}{2}\right) - i\frac{\omega_1 - \omega\cos\alpha}{\lambda}\sin\left(\frac{\lambda t}{2}\right)\right]\mathrm{e}^{-i\omega t/2}\chi_+(t) + i\left[\frac{\omega}{\lambda}\sin\alpha\sin\left(\frac{\lambda t}{2}\right)\right]\mathrm{e}^{+i\omega t/2}\chi_-(t). \tag{3}$$

(c) Compute the probability of a transition to spin down exactly.

Consider then the two regimes:

- $\omega \ll \omega_1$ (the so-called *adiabatic regime*)
- $\omega \gg \omega_1$

and compare the two behaviors. Is the adiabatic theorem satisfied?

(d) If $\omega_1 \ll \omega$, $H(t)$ can be considered as a time-dependent perturbation. Assuming that the perturbation is switched on at $t > 0$, compute the first order perturbative correction to the eigenstate

$$\chi(0) = \begin{pmatrix} \cos(\alpha/2) \\ \sin(\alpha/2) \end{pmatrix}$$

in time-dependent perturbation theory and compare it to the expansion of the exact solution (3).

(e) Consider now again the adiabatic regime $\omega \ll \omega_1$. Starting from the exact solution (3), show that in this regime it can be written as:

$$\chi(t) = \mathrm{e}^{i(\,\epsilon(t)\,+\,\gamma(t)\,)}\chi_+(t) + \mathcal{O}(\omega/\omega_1)$$

where $\epsilon(t)$ is the so-called dynamic phase, and $\gamma(t)$ is the geometric phase.

(f) The geometric phase can also be computed directly from $\chi_+(t)$, using the adiabatic approximation. Compute the geometric phase and compare the result with what you obtained in point (e).