# MMP I
# Exercise Sheet 11

HS 21
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**Exercise 1** [Schrödinger equation (6 points)]

Calculate the first eigenvalue $\lambda_1^{(a)}$ of the 1-dimensional Schrödinger equation:

$$u''(x) + [\lambda - W(x)]\, u(x) = 0$$

under the condition $u(-\infty) = 0 = u(+\infty)$, where

$$W(x) = \begin{cases} 0 & \text{for } |x| \geq a \\ -K/(2a) & \text{for } -a < x < a \end{cases}$$

is the potential (with $K, a > 0$).

a) Which "trascendent" equation identifies $\lambda_1^{(a)}$?
*Hint:* Solve the differential equation in the three regions $x > a$, $x < -a$ and $-a < x < a$. Notice that we expect $u(-x) = u(x)$, why? Use the continuity of the solutions and their first derivatives in $x = a$ and/or $x = -a$ to obtain an equation for $\lambda$.

b) Show, using the Rayleigh's principle, that $\lambda_1^{(a)}$ is an increasing function of $a$, i.e. if $a < a^+ \to \lambda_1^{(a)} \leq \lambda_1^{(a^+)}$. *Hint*: if $f(x) > 0$ and decreasing in $x$, $\frac{1}{l}\int_0^l f^2(x)dx \geq \frac{1}{L}\int_0^L f^2(x)dx$ for $L \geq l$.

– please turn over –

**Exercise 2** [Homogeneous string (6 points)]

The eigenvalue problem of a homogeneous vibrating string of length 1 ($0 \le x \le 1$) with the left endpoint fixed and the right endpoint free consists of minimizing the functional

$$S[f] = \int_0^1 [f'(x)]^2 \, dx \tag{1}$$

with $f(0) = 0$ and $\int_0^1 [f(x)]^2 \, dx = 1$. In this case we don't have a boundary condition for $x = 1$, so we must choose a natural constraint (Natürliche Randbedingung) in that point.

(a) [0.5 points] Compute the variation of the functional $\delta S = S[f+\delta f]-S[f]$ for an arbitrary variation $\delta f$ consistent with the given constraints and find the natural constraint at $x = 1$.
*Hint: This corrsponds to making the variational problem independent of the variation which we choose.*

(b) [0.5 points] Show that for this problem with our boundary condition for $x = 0$ and the natural constraint there exists a hermitian operator $A$ such that

$$(Af, f) = \int_0^1 [f'(x)]^2 \, dx \tag{2}$$

Determine the hermitian operator $A$ for this problem. Then, solving the vibrating string is equivalent to finding the spectrum of the Hermitian operator A.

(c) *Applying test functions to R[f]* [1.5 points]
In terms of the operator A, the variational formulation of the problem can be recast in the form of the minimisation of the corresponding Rayleigh quotient

$$R[f] = \frac{(Af, f)}{(f, f)} = \frac{\int_0^1 [f'(x)]^2 \, dx}{\int_0^1 [f(x)]^2 \, dx}. \tag{3}$$

Plug the test functions

$$g_1(x) = x, \qquad g_2(x) = -\frac{x^3}{6} + \frac{x}{2}, \qquad g_3(x) = +\frac{x^5}{120} - \frac{x^3}{12} + \frac{5}{24}x$$

into $R[\,\cdot\,]$. Can you give an upper boundary for the lowest eigenvalue of $A$?

(d) [3.5 points] Determine the eigenvalues and eigenfunctions for $A$ by making the Rayleigh quotient stationary, by introducing a Lagrange multiplier like in the lecture, and solve the ensuing Euler-Lagrange equation under the constraints of the problem. Which is the lowest eigenvalue?
*Hint: You don't have to normalize the resulting eigenfunctions of the problem.*