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Applications of General Relativity

Solutions 6

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Exercise 1 [Non-monochromatic gravitational waves]

We start from Eq. (5.54) in the script.

$$\begin{aligned} R^{(2)}_{\mu\kappa} =& -\frac{h^{\lambda\nu}}{2}\left[\frac{\partial^2 h_{\lambda\nu}}{\partial x^\mu \partial x^\kappa} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x^\kappa} - \frac{\partial^2 h_{\lambda\kappa}}{\partial x^\mu \partial x^\nu}\right] + \\ & + \frac{1}{4}\left[\frac{\partial h^\nu{}_\sigma}{\partial x^\nu} + \frac{\partial h^\nu{}_\sigma}{\partial x^\nu} - \frac{\partial h^\nu{}_\nu}{\partial x^\sigma}\right]\left[\frac{\partial h^\sigma{}_\mu}{\partial x^\kappa} + \frac{\partial h^\sigma{}_\kappa}{\partial x^\mu} - \frac{\partial h_{\mu\kappa}}{\partial x_\sigma}\right] - \\ & - \frac{1}{4}\left[\frac{\partial h_{\sigma\kappa}}{\partial x^\lambda} + \frac{\partial h_{\sigma\lambda}}{\partial x^\kappa} - \frac{\partial h_{\lambda\kappa}}{\partial x^\sigma}\right]\left[\frac{\partial h^\sigma{}_\mu}{\partial x_\lambda} + \frac{\partial h^{\sigma\lambda}}{\partial x^\mu} - \frac{\partial h^\lambda{}_\mu}{\partial x_\sigma}\right]. \end{aligned} \tag{1}$$

We will now pick a gauge in which $h_{\alpha\beta}$ is transverse $\partial_\mu h^{\mu\nu} = 0$ and traceless $h^\mu_\mu = 0$. Thus the trace reverses metric and the metric agree $h_{\mu\nu} = h_{\mu\nu}$ and we will use h from now on. In this gauge the first bracket in the second line vanishes automatically. For the remaining terms we have to perform partial integrations neglecting the surface terms

$$\begin{aligned} & h^{\lambda\nu}\left[\frac{\partial^2 h_{\lambda\nu}}{\partial x^\mu \partial x^\kappa} + \frac{\partial^2 h_{\mu\kappa}}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x^\kappa} - \frac{\partial^2 h_{\lambda\kappa}}{\partial x^\mu \partial x^\nu}\right] \\ =& -\frac{\partial h^{\lambda\nu}}{\partial x^\mu}\frac{\partial h_{\lambda\nu}}{\partial x^\kappa} - \frac{\partial h^{\lambda\nu}}{\partial x^\lambda}\frac{\partial h_{\mu\kappa}}{\partial x^\nu} + \frac{\partial h^{\lambda\nu}}{\partial x^\lambda}\frac{\partial h_{\mu\nu}}{\partial x^\kappa} + \frac{\partial h^{\lambda\nu}}{\partial x^\nu}\frac{\partial h_{\lambda\kappa}}{\partial x^\mu} \\ =& -\partial_\mu h^{\lambda\nu}\partial_\kappa h_{\lambda\nu} \end{aligned} \tag{2}$$

Except for the first one, all the terms vanish because h is transverse. It remains to manipulate the third line in Eq. (1), again integrating by parts and neglecting boundary terms

$$\begin{aligned} & \left[\frac{\partial h_{\sigma\kappa}}{\partial x^\lambda} + \frac{\partial h_{\sigma\lambda}}{\partial x^\kappa} - \frac{\partial h_{\lambda\kappa}}{\partial x^\sigma}\right]\left[\frac{\partial h^\sigma{}_\mu}{\partial x_\lambda} + \frac{\partial h^{\sigma\lambda}}{\partial x^\mu} - \frac{\partial h^\lambda{}_\mu}{\partial x_\sigma}\right] \\ =& -h_{\sigma\kappa}\partial_\lambda\partial^\lambda h^\sigma{}_\mu - h_{\sigma\kappa}\partial_\mu\partial_\lambda h^{\sigma\lambda} + h_{\sigma\kappa}\partial^\sigma\partial_\lambda h^\lambda{}_\mu \\ & -\partial_\kappa\partial^\lambda h_{\sigma\lambda}h^\sigma{}_\mu + \partial_\kappa h_{\sigma\lambda}\partial_\mu h^{\sigma\lambda} + \partial_\kappa\partial^\sigma h_{\sigma\lambda}h^\lambda{}_\mu \\ & + h_{\lambda\kappa}\partial^\lambda\partial_\sigma h^\sigma{}_\mu + h_{\lambda\kappa}\partial_\mu\partial\sigma h^{\sigma\lambda} - h_\lambda\kappa\partial_\sigma\partial^\sigma h^\lambda{}_\mu \\ =& \partial_\kappa h_{\sigma\lambda}\partial_\mu h^{\sigma\lambda} \end{aligned} \tag{3}$$

With the wave equation $\Box h_{\mu\nu} = 0$ and the transversality condition one can cancel all the terms except for the second term in the second line.

$$\left\langle R^{(2)}_{\mu\kappa}\right\rangle = \frac{1}{4}\partial_\mu h^{\lambda\nu}\partial_\kappa h_{\lambda\nu} \tag{4}$$

Note that

$$\left\langle R^{(2)}\right\rangle = \frac{1}{4}\left\langle \partial^\kappa h^{\lambda\nu}\partial_\kappa h_{\lambda\nu}\right\rangle = -\frac{1}{4}\left\langle h^{\lambda\nu}\partial^\kappa\partial_\kappa h_{\lambda\nu}\right\rangle = 0 \tag{5}$$

due to the wave equation. From the lecture we have

$$t^{\mu\nu}_{\text{grav}} = \frac{c^4}{16\pi G}\left[2R^{(2)}_{\mu\nu} - \eta_{\mu\nu}\eta^{\rho\sigma}R^{(2)}_{\rho\sigma}\right] \tag{6}$$

Thus

$$t^{\text{grav}}_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h^{\alpha\beta}\partial_\nu h_{\alpha\beta}\right\rangle \tag{7}$$

Using the three remaining gauge modes we can set $h^{i0} = 0$. From the transversality condition we have for the time component

$$\partial_0 h^{00} = \partial_i h^{i0} = 0. \tag{8}$$

For gravitational waves we are interested in the dynamical degrees of freedom and not in constant metric perturbations corresponding to the Newtonian potential, thus we can set $h^{00} = 0$ and are left with a purely spatial metric perturbation.

$$t^{\text{grav}}_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle \partial_\mu h^{ij}\partial_\nu h_{ij}\right\rangle. \tag{9}$$

The time-time component reads as

$$t^{\text{grav}}_{00} = \frac{c^4}{32\pi G}\left\langle \partial_0 h^{ij}\partial_0 h_{ij}\right\rangle \tag{10}$$

and for plane wave perturbations $h_{\mu\nu} = e_{\mu\nu}\exp\left[-ik_\lambda x^\lambda\right] + \text{c.c.}$ it simplifies to

$$t^{\text{grav}}_{00} = \frac{c^4\omega^2}{16\pi G}e^{ij}e_{ij}, \tag{11}$$

which is in accordance to the result presented in the lecture.

Exercise 2 [Plugging in the numbers]

(i) We have $\omega(t=0)$, and $m_1 = m_2 = m \Rightarrow 2\mathcal{M}^5 = m^5$, and we want to compute t_0 :

$$\omega(0)^{-8} = 2^{21}(G\mathcal{M})^5 t_0^3 \Rightarrow t_0 = 2^{-20/3}c^5(Gm)^{-5/3}\omega(0)^{-8/3} \approx 2\times 10^{14}\text{s} \approx 7\times 10^6 \text{ yr} \tag{12}$$

(ii) As the two black holes are merging, we are in the limiting case where the Newtonian approximation breaks down. This is typically the case when the two black holes are separated by a distance comparable to their Schwarzschild radius, i.e. $R \approx GM/c^2$.

(a) The frequency of the gravitational waves is approximately:

$$\nu = \frac{\omega}{2\pi} = \sqrt{\frac{GM}{R^3}} = \frac{c^3}{GM} = 10^{-4}\text{ Hz} \tag{13}$$

(b) The dimensionless strain, defined as the relative amplitude of the oscillation of a ruler when a gravitational wave passes through, is:

$$h = |h_{ij}| = \frac{GM}{c^2|x|} = 10^{-14} \tag{14}$$

(c) We take here the formula from part a), with the frequency of part i), dropping all constant terms and find:

$$t_0 = \frac{GM}{c^3} = 5000\text{ s} = 1\text{ h} \tag{15}$$