



Mechanics

Exercise Sheet 5

HS 2026
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Exercise 1: Three point particles

The system of Fig. 1 consists of three point particles, P_1 and P_2 of mass m and G of mass $2m$. All three points move frictionlessly in the $y = 0$ plane, and G is constrained to stay in the center of mass of the P_1, P_2 system. P_1 and P_2 interact by means of a conservative force with potential energy

$$V_{\text{int}}(\mathbf{r}_1, \mathbf{r}_2) = a|\mathbf{r}_1 - \mathbf{r}_2|^3 - b|\mathbf{r}_1 - \mathbf{r}_2|^2 \quad (1.1)$$

with $a, b > 0$. G is connected to the origin O by means of a spring of zero rest length and spring constant k . All three points are subject to gravity ($\mathbf{g} = -g\mathbf{e}_z$).

- a) How many generalised coordinates are required to describe the system, and why? Write the Euler-Lagrange equations of the system, using as free coordinates the x, z coordinate of G , the distance r of P_1 from G and the angle θ between $\mathbf{r}_1 - \mathbf{r}_G$ and the x -axis, as in Fig.1.
- b) Write the explicit expressions of *i)* the total energy, *ii)* the y -component of the total angular momentum of the system computed with respect to G and *iii)* the x -component of the total momentum of the system.
Are these quantities conserved? State why/why not on the basis of the Lagrangian symmetries. Do your conclusions change if $k = 0$?
- c) Find the equilibrium configurations, i.e. configurations x_0, z_0, r_0, θ_0 such that $\ddot{q} = \dot{q} = 0$, disregarding the trivial case $r_0 = 0$. Show that they are all unstable.
Hint: Study the behaviour of perturbations around θ_0 .

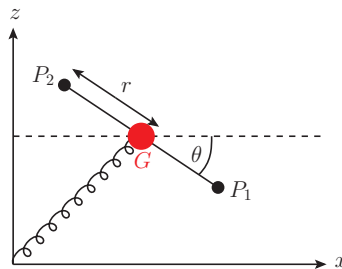


Figure 1: System of exercise 1.

Solution:

- a) In absence of constraints the system has 9 degrees of freedom. The requirements $y = 0$ and $\mathbf{r}_G = \mathbf{r}_{CM} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ impose 5 conditions, thus reducing the number of dynamical degrees of freedom to 4. The motion of the system can therefore be described in terms of four generalized coordinates. In the following we will use the coordinates suggested in point a), namely x , z , r and θ .

Since P_1 and P_2 have the same mass, their center of mass is situated exactly in the midpoint of segment P_1P_2 . Using the coordinates above, the positions and velocities of the three particles can be written as

$$\mathbf{r}_G = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}, \quad \mathbf{r}_1 = \begin{pmatrix} x + r \cos \theta \\ 0 \\ z - r \sin \theta \end{pmatrix}, \quad \mathbf{r}_2 = \begin{pmatrix} x - r \cos \theta \\ 0 \\ z + r \sin \theta \end{pmatrix}, \quad (1.2)$$

$$\mathbf{v}_G = \begin{pmatrix} \dot{x} \\ 0 \\ \dot{z} \end{pmatrix}, \quad \mathbf{v}_1 = \begin{pmatrix} \dot{x} + \dot{r} \cos \theta - r \dot{\theta} \sin \theta \\ 0 \\ \dot{z} - \dot{r} \sin \theta - r \dot{\theta} \cos \theta \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} \dot{x} - \dot{r} \cos \theta + r \dot{\theta} \sin \theta \\ 0 \\ \dot{z} + \dot{r} \sin \theta + r \dot{\theta} \cos \theta \end{pmatrix}. \quad (1.3)$$

The kinetic and potential energy are therefore given by

$$T = m\mathbf{v}_G^2 + \frac{1}{2}m(\mathbf{v}_1^2 + \mathbf{v}_2^2) = 2m(\dot{x}^2 + \dot{z}^2) + m(\dot{r}^2 + r^2\dot{\theta}^2), \quad (1.4)$$

$$V = 8ar^3 - 4br^2 + \frac{k}{2}(x^2 + z^2) + 4mgz,$$

yielding the Lagrangian

$$L = T - V = 2m(\dot{x}^2 + \dot{z}^2) + m(\dot{r}^2 + r^2\dot{\theta}^2) - 8ar^3 + 4br^2 - \frac{k}{2}(x^2 + z^2) - 4mgz, \quad (1.5)$$

so that the Euler-Lagrange equations of the system are

$$\begin{aligned} \ddot{x} &= -\frac{k}{4m}x, \\ \ddot{z} &= -\frac{k}{4m}z - g, \\ \ddot{r} &= r\dot{\theta}^2 - \frac{12a}{m}r^2 + \frac{4b}{m}, \\ \frac{d}{dt}(2mr^2\dot{\theta}) &= 0. \end{aligned} \quad (1.6)$$

- b) The total energy, the y -component of the total angular momentum around G and the x -component of the total momentum are

$$\begin{aligned} E &= T + V = 2m(\dot{x}^2 + \dot{z}^2) + m(\dot{r}^2 + r^2\dot{\theta}^2) + 8ar^3 - 4br^2 + k/2(x^2 + z^2) + 4mgz, \\ L_y|_G &= (m(\mathbf{r}_1 - \mathbf{r}_G) \times \mathbf{v}_1 + m(\mathbf{r}_2 - \mathbf{r}_G) \times \mathbf{v}_2)_y = 2mr^2\dot{\theta}, \\ p_x &= (m\mathbf{v}_1 + m\mathbf{v}_2 + 2m\mathbf{v}_G)_x = 4m\dot{x}. \end{aligned}$$

The Lagrangian does not depend explicitly on time, hence the total energy E is conserved. $L_y|_G$ is also conserved, being the conjugate momentum of the cyclic coordinate θ :

$$\frac{\partial L}{\partial \theta} = 0 \quad \rightarrow \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = 2mr^2\dot{\theta} \text{ is conserved} \quad (1.7)$$

This corresponds to the fact that the system is invariant under rotations around the point G in the $y = 0$ plane. Note that the conservation of $L_y|_G$ is manifest already in eq. 1.6.

The x -component of momentum instead is not conserved, because the Lagrangian is not invariant under translations along the x -axis. If $k = 0$ x becomes a cyclic variable, hence $\frac{\partial L}{\partial \dot{x}} = 4m\dot{x} = p_x$ is also conserved.

- c) To find the equilibrium configurations we impose $\ddot{q} = \dot{q} = 0$ for $q = x, z, r, \theta$ in the Euler-Lagrange equations. Disregarding the trivial solution $r_0 = 0$, we find

$$x_0 = 0, \quad z_0 = -\frac{4mg}{k}, \quad r_0 = \frac{b}{3a}, \quad \theta_0 \in \mathbb{R}. \quad (1.8)$$

We can easily show that these configurations are all unstable. Consider a small perturbation around θ_0 and $\dot{\theta} = 0$: $\theta(t) = \theta_0 + \alpha(t)$. From the equation of motion for θ we find

$$r^2 \ddot{\alpha} = \text{const.} \quad \rightarrow \quad \alpha(t) \sim ct + d \quad (1.9)$$

i.e. a perturbation grows linearly in time.

Exercise 2: Circular orbits of the Yukawa potential. (3+2 points)

Consider a particle of mass m moving in the central potential

$$V = -k \frac{e^{-ar}}{r} \quad (2.1)$$

with the constants $k > 0$ and $a > 0$.

- a) In analogy to the Kepler problem, study the effective potential and prove:

$$\text{Stable circular orbits exist if } r < \frac{(1 + \sqrt{5})}{2a}. \quad (2.2)$$

- b) Study the circular orbits for small values of r by expanding the minimizing condition of the effective potential around $r = 0$ up to (and including) $\mathcal{O}(r^3)$. Which values of ℓ^2 (with ℓ being the angular momentum) do the circular orbits take in terms of r and the given constants a, k, m ? Compare your result to the values found for a gravitational potential.

Solution:

- a) The effective potential, is given by

$$V_{\text{eff}} = \frac{\ell^2}{2mr^2} + V = \frac{\ell^2}{2mr^2} - k \frac{e^{-ar}}{r}. \quad (2.3)$$

A stable circular orbit can only exist if the effective potential has a minimum. Extremal points of this effective potential satisfy

$$\frac{dV_{\text{eff}}}{dr} = 0. \quad (2.4)$$

So we compute:

$$\begin{aligned} 0 &= \frac{dV_{\text{eff}}}{dr} = -\frac{\ell^2}{mr^3} + \frac{k}{r^2} e^{-ar} + \frac{ak}{r} e^{-ar} = -\frac{\ell^2}{mr^3} + \frac{k}{r^2} (1 + ar) e^{-ar} \\ \implies \frac{\ell^2}{m} &= e^{-ar} k r (1 + ar). \end{aligned} \quad (2.5)$$

This always has a solution since the right-hand side is always larger than zero. However, we need to verify that the solution is a minimum of the potential by checking its curvature

$$\left. \frac{d^2 V_{\text{eff}}}{dr^2} \right|_{r=r_{\text{ext}}} > 0. \quad (2.6)$$

And so we compute the second derivative:

$$\begin{aligned} \frac{d^2 V_{\text{eff}}}{dr^2} &= \frac{3\ell^2}{mr^4} - \frac{2k}{r^3} e^{-ar} - 2 \frac{ak}{r^2} e^{-ar} - \frac{a^2 k}{r} e^{-ar} \\ &= 3 \frac{\ell^2}{mr^4} - e^{-ar} \left(2 \frac{k}{r^3} + 2 \frac{ak}{r^2} + \frac{a^2 k}{r} \right). \end{aligned} \quad (2.7)$$

We are evaluating this expression at the extremal point, so we can use eq. (2.5) to replace ℓ^2 . The result reads:

$$\begin{aligned} \left. \frac{d^2 V_{\text{eff}}}{dr^2} \right|_{r=r_{\text{ext}}} &= 3 \frac{k}{r^3} (1 + ar) e^{-ar} - e^{-ar} \left(2 \frac{k}{r^3} + 2 \frac{ak}{r^2} + \frac{a^2 k}{r} \right) \\ &= e^{-ar} \frac{k}{r} \left(\frac{1}{r^2} + \frac{a}{r} - a^2 \right) \\ &= e^{-ar} \frac{k}{r^3} (1 + ar - a^2 r^2). \end{aligned} \quad (2.8)$$

Now imposing that this expression shall be positive yields:

$$a^2 r^2 - 1 - a r < 0. \quad (2.9)$$

The cross-over values for r lie at

$$r_{1/2} = \frac{1 \pm \sqrt{5}}{2a}, \quad (2.10)$$

where only the sign choice with $+$ yields a positive value and returns the condition for stable orbits to be:

$$r < \frac{1 + \sqrt{5}}{2a}, \quad (2.11)$$

which is what we wanted to show.

b) We start at the condition (2.5) and expand it around small values of r :

$$\begin{aligned} \frac{\ell^2}{m} &= e^{-ar} k r (1 + a r) \\ &= \left(1 - ar + \frac{a^2 r^2}{2} + \mathcal{O}(r^3) \right) k r (1 + a r) \\ &= k r - \frac{1}{2} a^2 k r^3 + \mathcal{O}(r^4) \\ &= k r \left(1 - \frac{a^2 r^2}{2} + \mathcal{O}(r^3) \right). \end{aligned} \quad (2.12)$$

Notice that the leading term is precisely the value for the Newtonian gravitational potential.