MMP I
Exercise Sheet 9

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https://www.physik.uzh.ch/en/teaching/PHY312

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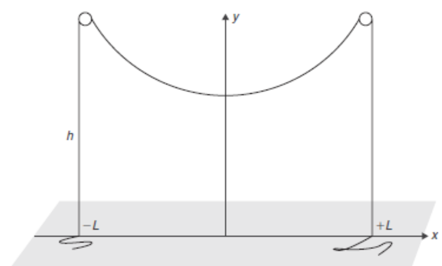
Exercise 1 [Hanging chain (6 points)]

An ideal wire of length $2l$ and constant specific weight μ is fixed at the points $x = -L$ and $x = L$ at the same height h. Which is its shape?



a) 1. Write the potential energy $V[y]$ of this system as an integral over x.
   2. Write down the length condition $G[y] = 2l$ as an integral over x as well.
   3. Insert a Lagrange multiplier λ and minimize the function $V - \lambda G$.

   Hint:

   - Use the first integral of the Euler equation derived in the lecture for the case $F_x = 0$.
   - For the integration of the differential equation use an appropriate substitution and the relation $\cosh^2 \varphi - \sinh^2 \varphi = 1$.

b) Use the symmetry of $y(x)$ to find the value of the additive constant which entered when you integrated the differential equation.

c) Use the length condition $G[y]$ to write down an equation for the constant which arose from the first integral. This equation can only be solved numerically, which you don't have to do.

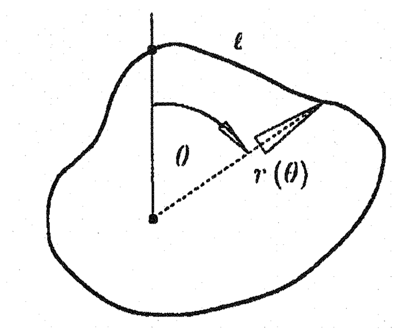
d) Evaluate the boundary conditions $y(-L) = y(L) = h$ to recover the value of λ as a function of the constant from c).

– please turn over –

Exercise 2 [Loop (3 points)]

With a piece of string of length L a loop around the point of origin ($r = 0$) is formed. Find the curve which maximizes the area of the loop. What shape does it form?

Hint: Work in polar coordinates and for an infinitesimal change in angle $d\theta$ find the corresponding area element dA and the line element dl.



Exercise 3 [Geometrical optics (5 points)]

In the lecture you have treated Fermat's principle for light rays in a dispersive medium.

a) Redo the derivation and show that the the eikonal equations are the Euler equation for the Fermat principle. Let $n(x, y, z)$ be the refraction index. From the Fermat principle we know that a ray of light, from a point A to a point B, follows a trajectory $\int_A^B n(x, y, z)ds$ that is stationary, i.e. minimizes the time-to-travel t. Verify that the result (the eikonal equation) can be written in the form:

$$\nabla n = \frac{\mathrm{d}}{\mathrm{d}s}\left(n\frac{\mathrm{d}\vec{x}}{\mathrm{d}s}\right)$$

b) On a plane: what first integral arises when $n(x, y) \equiv n(y)$?

c) Let us assume that $n(x, y) = \begin{cases} n_1, & y > 0 \\ n_2, & y < 0 \end{cases}$.

By assuming a natural boundary condition at $y = 0$, show that Snell's law ($n_1 \sin\alpha_1 = n_2 \sin\alpha_2$) holds. α shall depict the angle between the light ray and the y-axis.
Hint: Relate the different infinitesimals with α.