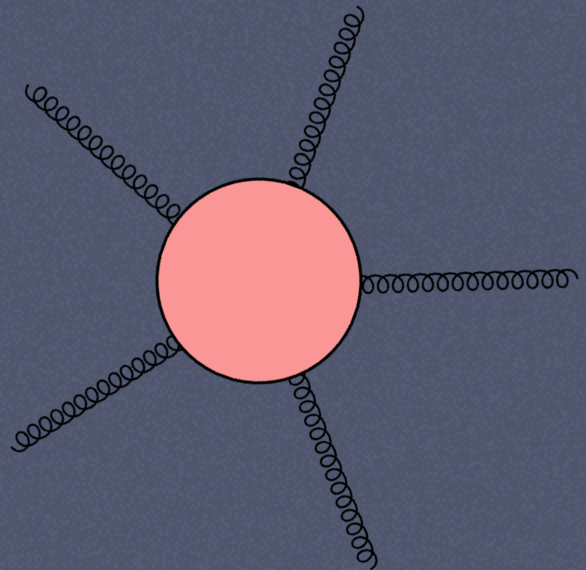
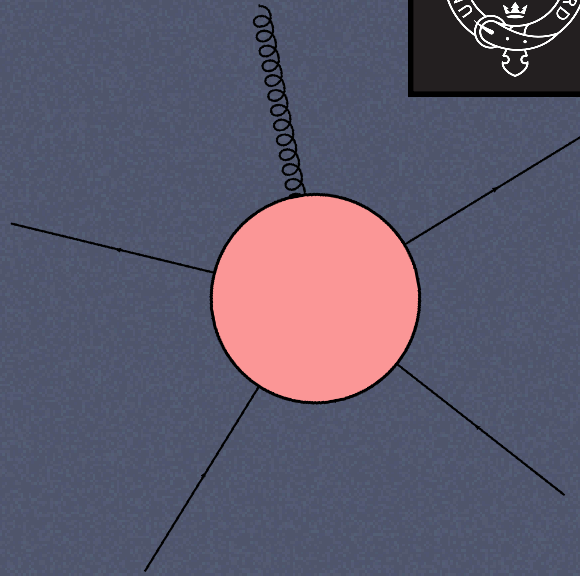
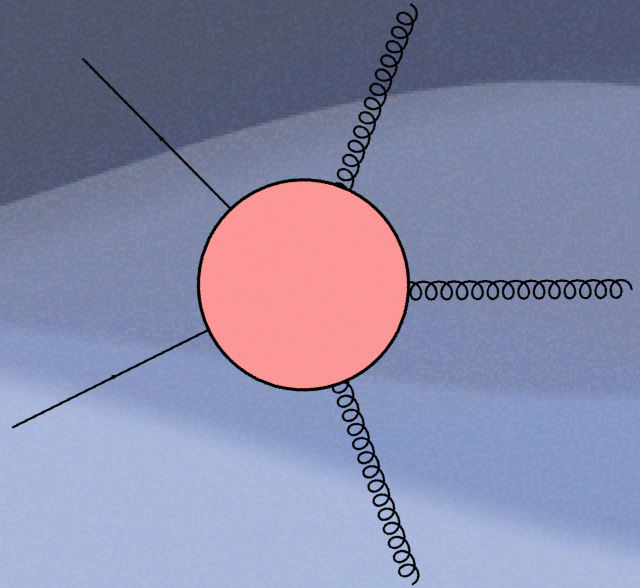




Two Loops & Five Partons

the Full Colour Story



In collaboration with: Bakul Agarwal
Federico Buccioni
Federica Devoto
Andreas von Manteuffel
Lorenzo Tancredi

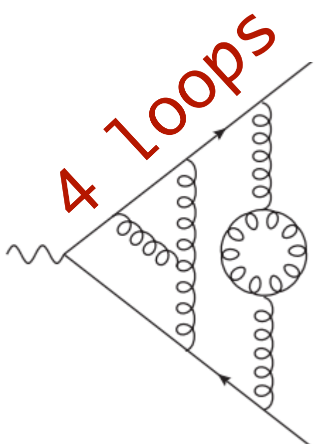
Giulio Gambuti - 24/10/2023 - ETH/UZH

- State of The Art
- Where 5-points fits
- The Computation
- Constraints on the Amplitudes

Massless-QCD: State of the Art

Massless-QCD: State of the Art

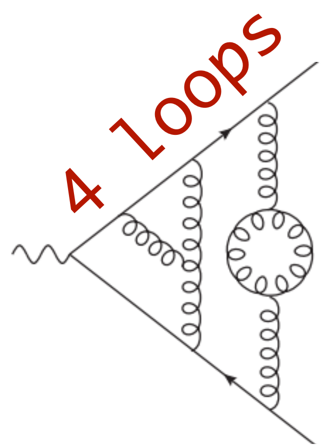
3-point



Lee, von Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser: 2202.04660(PRL)

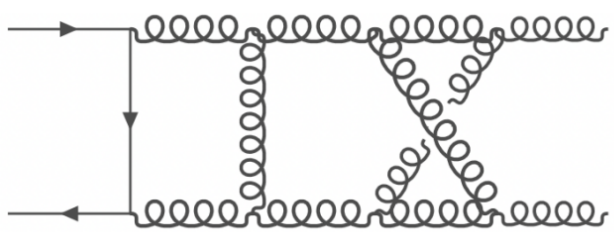
Massless-QCD: State of the Art

3-point



Lee, von Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser: 2202.04660(PRL)

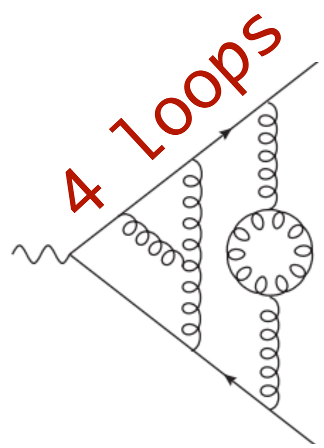
4-point



Caola, von Manteuffel, Tancredi: 2011.13946(PRL)
Bargiela, Caola, von Manteuffel, Tancredi:2111.13595(JHEP)
Chakraborty, Caola, GG, Tancredi, von Manteuffel: 2108.00055(JHEP), 2207.03503(JHEP), 2112.11097(PRL)
Bargiela, Chakraborty, GG: (2212.14069)

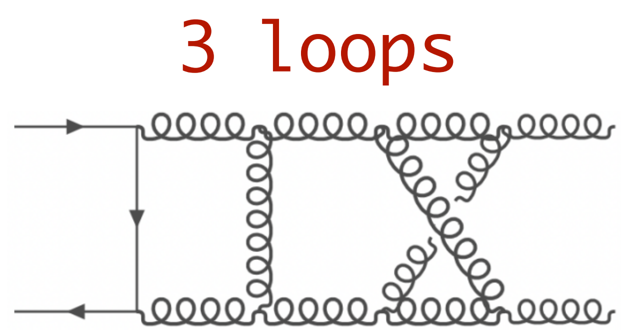
Massless-QCD: State of the Art

3-point



Lee, von Manteuffel, Schabinger, Smirnov, Smirnov,
Steinhauser: 2202.04660(PRL)

4-point



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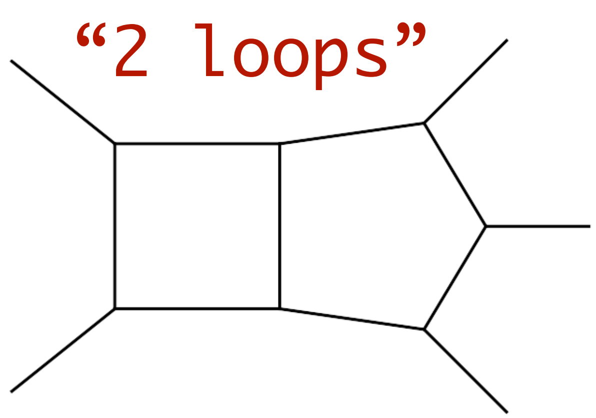
2108.00055(JHEP), 2207.03503(JHEP), 2112.11097(PRL)
Bargiela, Chakraborty, GG: (2212.14069)

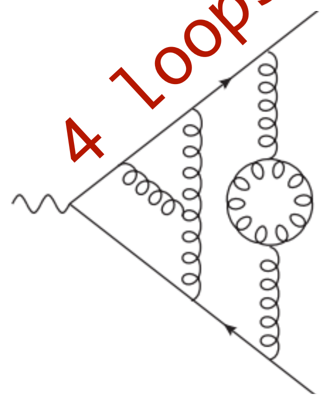
5-point



Abreu, Dormans, Cordero, Ita, Page: 1812.04586(PRL)
Badger, Gehrmann, Heinrich, Henn: 1905.03733(PRL)
Agrawal, Buccioni, von Manteuffel, Tancredi: 2105.04585(PRL)

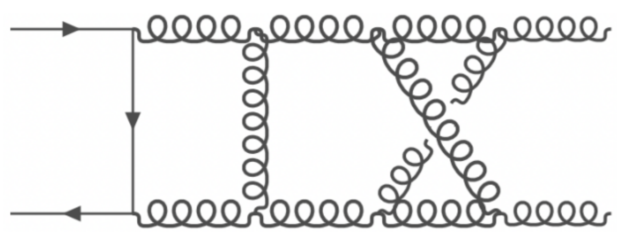
Massless-QCD: State of the Art

3-point — 4 loops



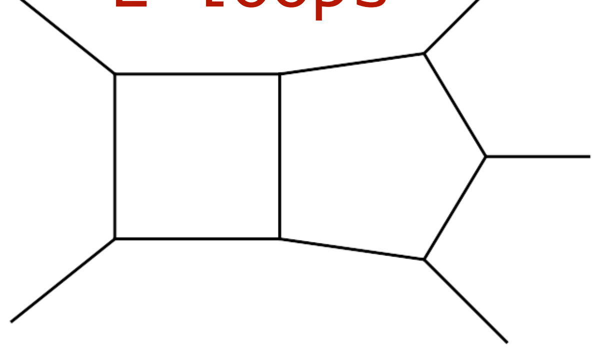
Lee, von Manteuffel, Schabinger, Smirnov, Smirnov, Steinhauser: 2202.04660(PRL)

4-point — 3 loops



Caola, von Manteuffel, Tancredi: 2011.13946(PRL)
Bargiela, Caola, von Manteuffel, Tancredi:2111.13595(JHEP)
Chakraborty, Caola, GG, Tancredi, von Manteuffel: 2108.00055(JHEP), 2207.03503(JHEP), 2112.11097(PRL)
Bargiela, Chakraborty, GG: (2212.14069)

5-point — “2 loops”



Abreu, Dormans, Cordero, Ita, Page: 1812.04586(PRL)
Badger, Gehrmann, Heinrich, Henn: 1905.03733(PRL)
Agrawal, Buccioni, von Manteuffel, Tancredi: 2105.04585(PRL)

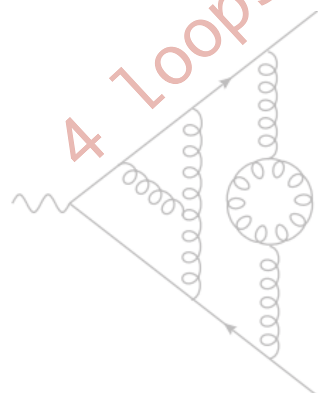
6-point — 1 loop …

Openloops 2: 1907.13071(EPJC)
Ellis, Giele, Zanderighi: 0602185(JHEP)

Massless-QCD: State of the Art

3-point

4 loops



Lee, von Manteuffel, Schabinger, Smirnov, Smirnov,
Steinhauser: 2202.04660(PRL)

4-point

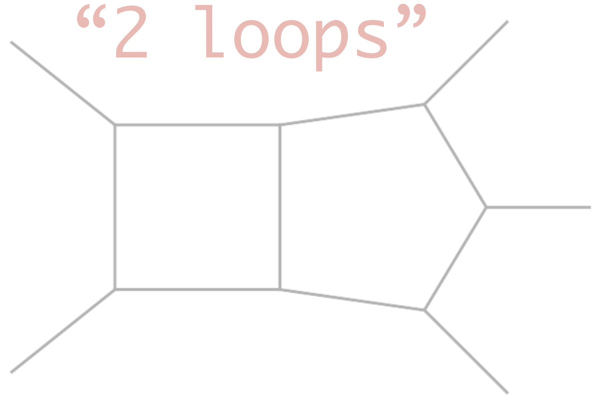
3 loops

Caola, von Manteuffel, Tancredi: 2011.13946(PRL)
Bargiela, Caola, von Manteuffel, Tancredi:2111.13595(JHEP)
Chakraborty, Caola, GG, Tancredi, von Manteuffel:
2108.00055(JHEP), 2207.03503(JHEP), 2112.11097(PRL)
Bargiela, Chakraborty, GG: (2212.14069)

Loops+Legs=7

5-point

“2 loops”



Abreu, Dormans, Cordero, Ita, Page: 1812.04586(PRL)
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6-point

1 loop …

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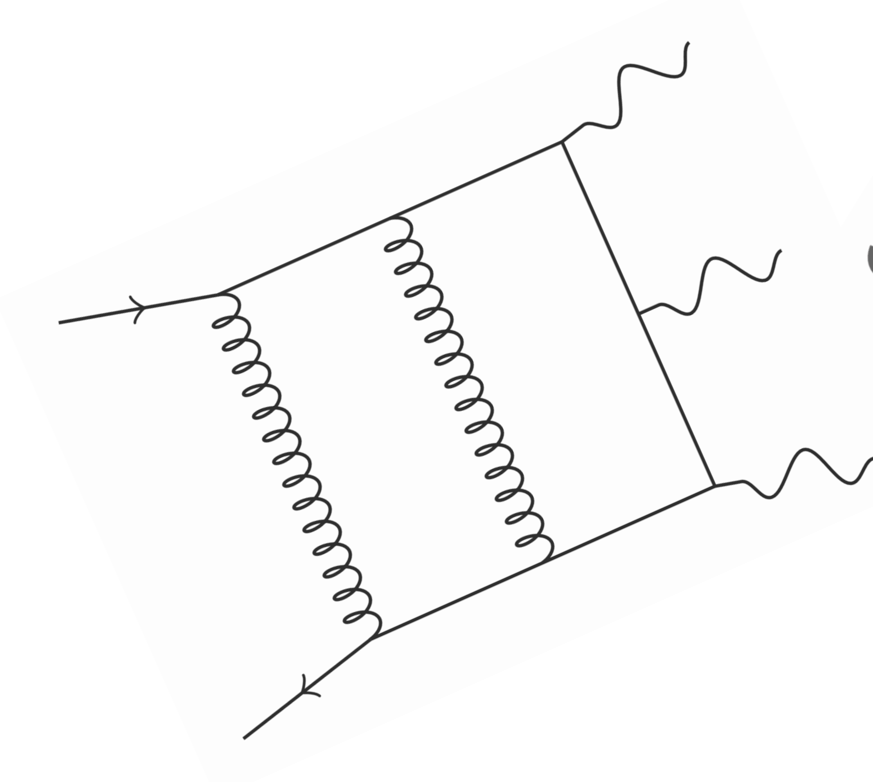
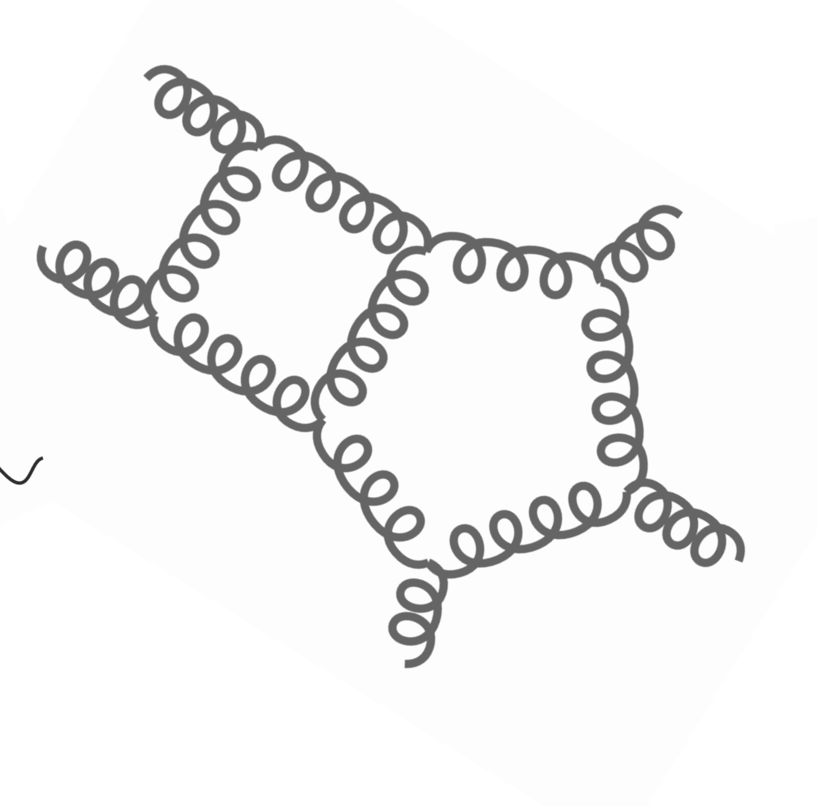
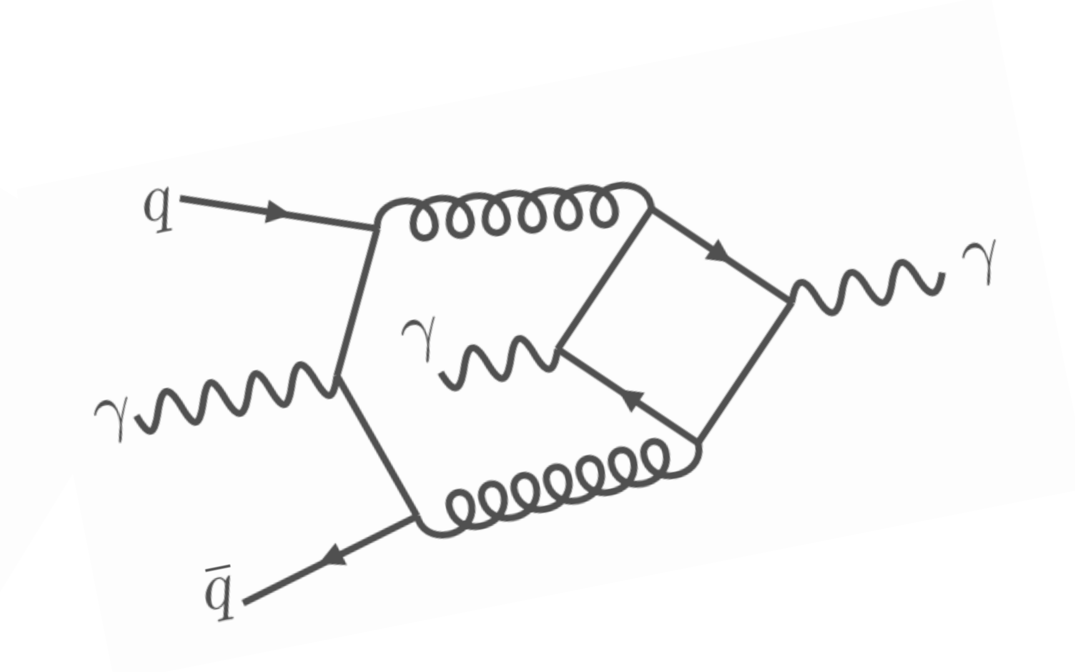






S. Badger, D. Chicherin, T. Gehrmann, G. Heinrich, J.M. Henn, T. Peraro, P. Wasser, Y. Zhang, S. Zoia: 1905.03733

Herschel A. Chawdhry, Michal Czakon, Alexander Mitov, Rene Poncelet: 2012.13553

Bakul Agarwal, Federico Buccioni, Andreas von Manteuffel, Lorenzo Tancredi: 2102.01820

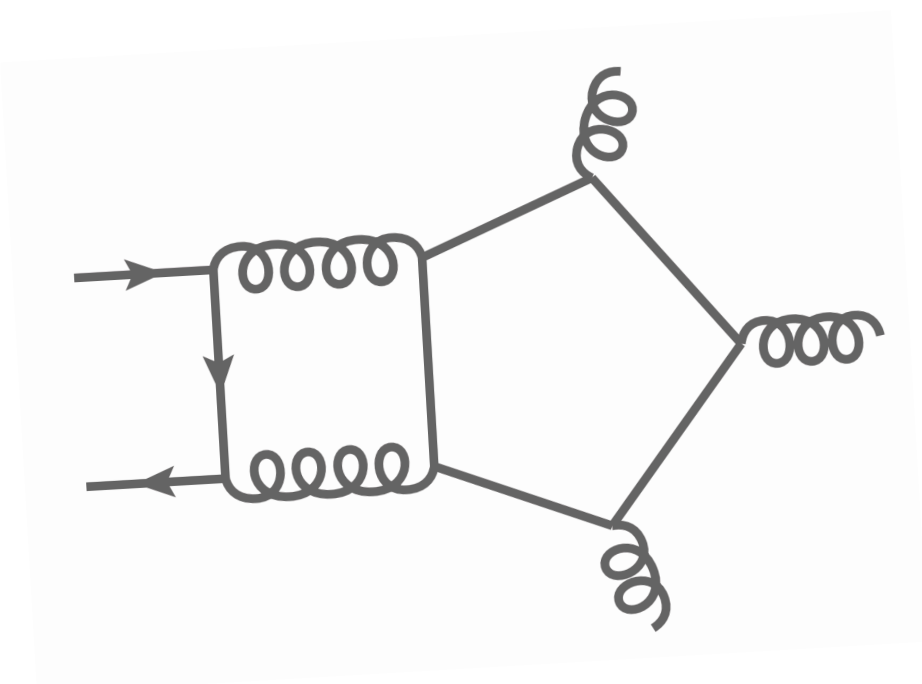
S. Abreu, F. Febres Cordero, H. Ita, B. Page, V. Sotnikov: 2102.13609

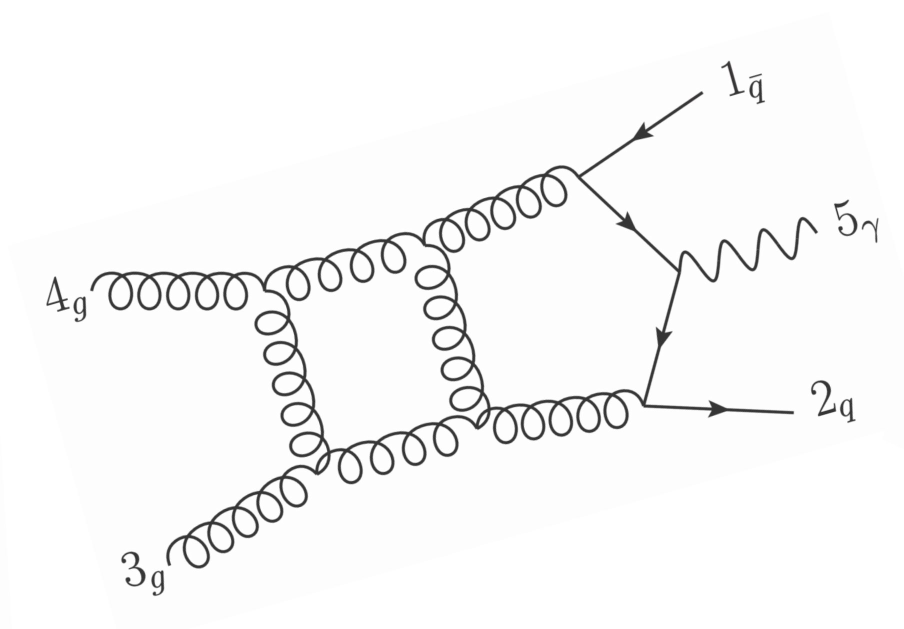
Bakul Agarwal, Federico Buccioni, Andreas von Manteuffel, Lorenzo Tancredi: 2105.04585

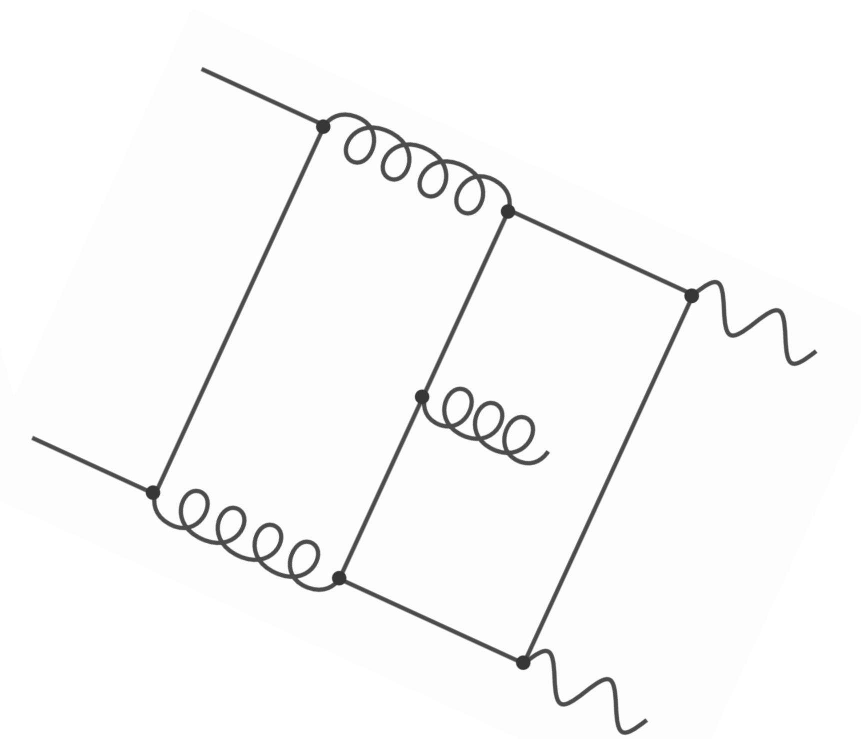
Michal Czakon, Alexander Mitov, Rene Poncelet: 2106.05331

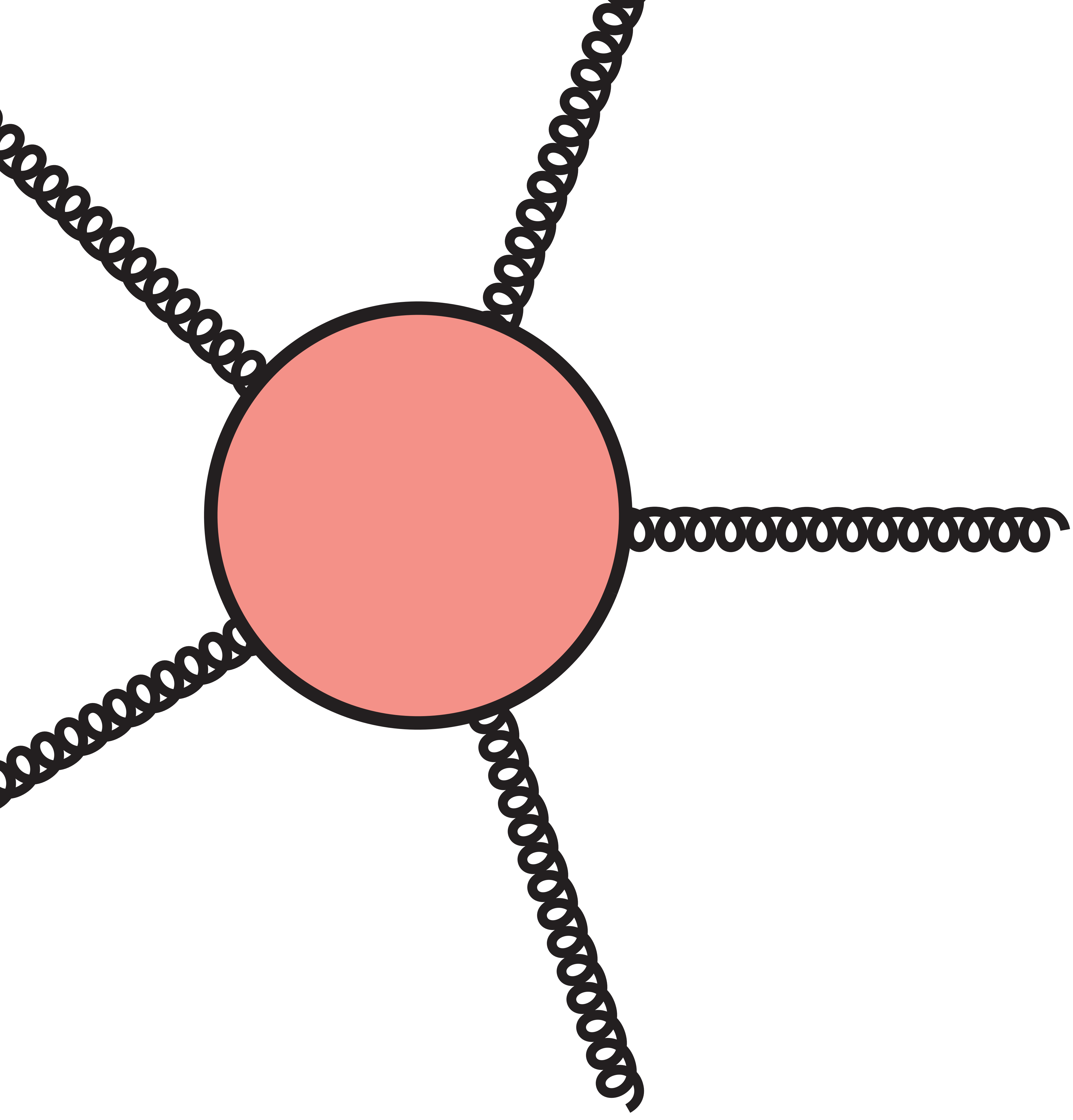
Simon Badger, Michał Czakon, Heribertus Bayu Hartanto, Ryan Moodie, Tiziano Peraro: 2304.06682

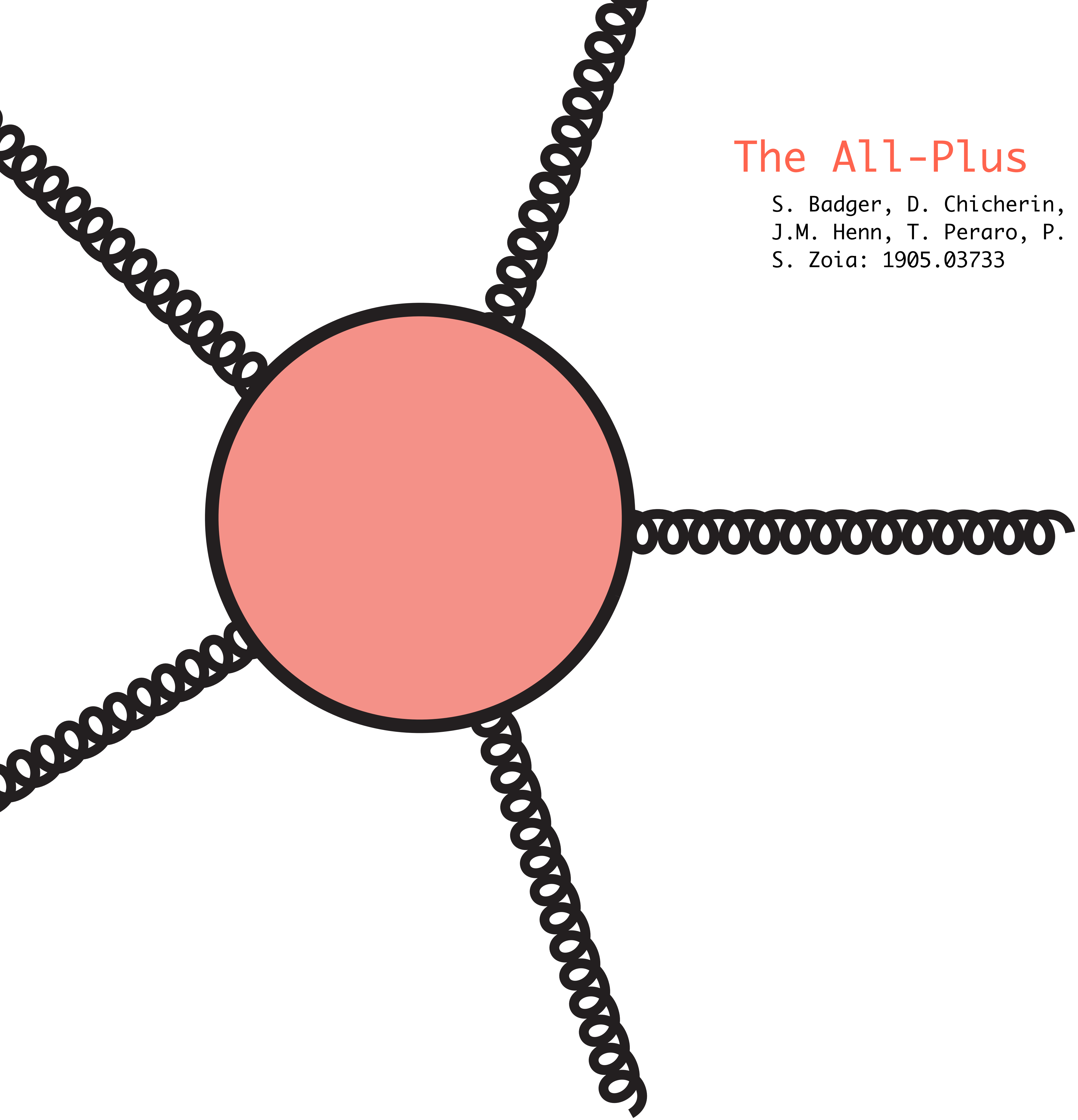
Samuel Abreu, Giuseppe De Laurentis, Harald Ita, Maximillian Klinkert, Ben Page, Vasily Sotnikov: 2305.17056





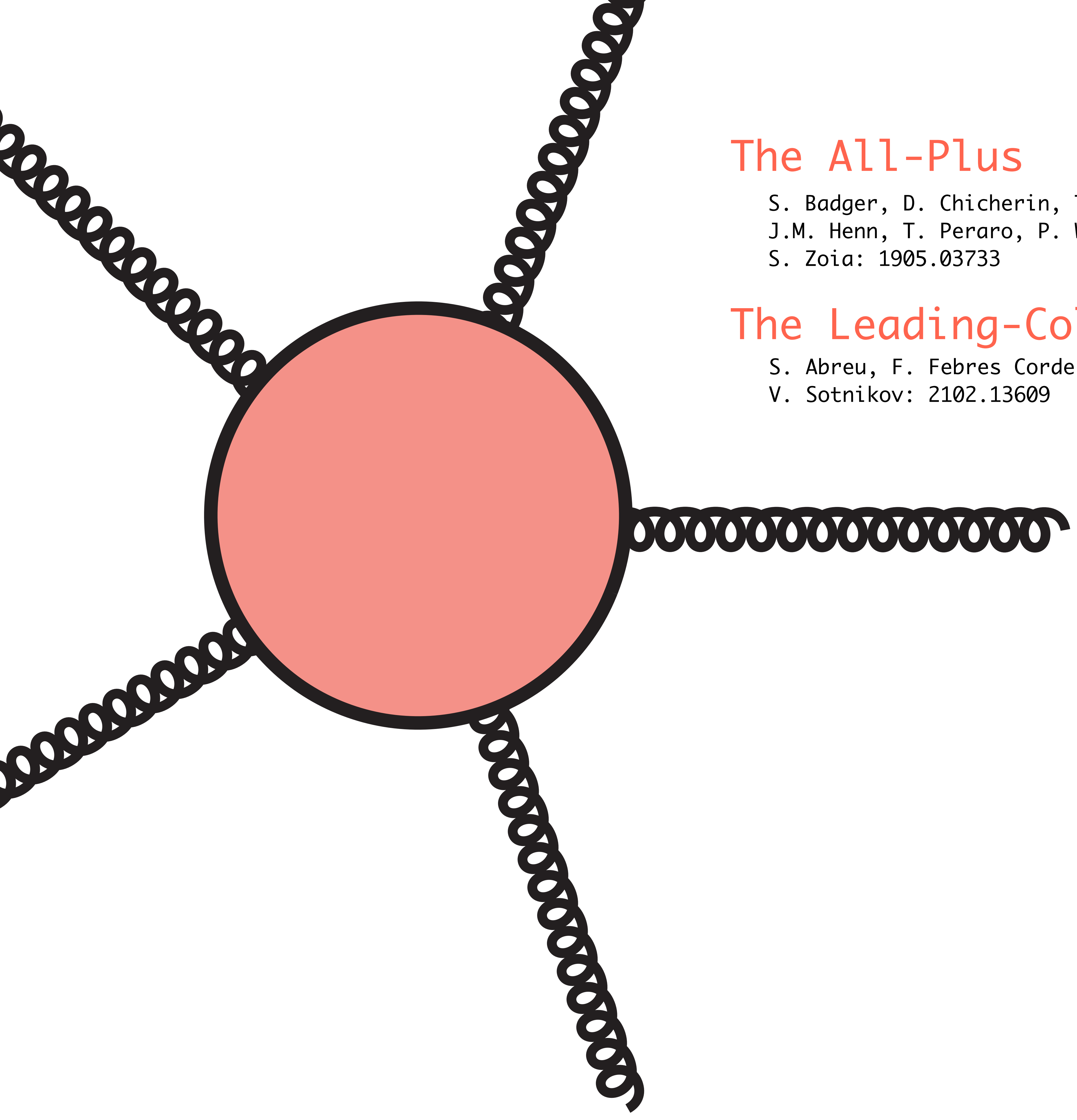






The All-Plus

S. Badger, D. Chicherin, T. Gehrmann, G. Heinrich, J.M. Henn, T. Peraro, P. Wasser, Y. Zhang, S. Zoia: 1905.03733

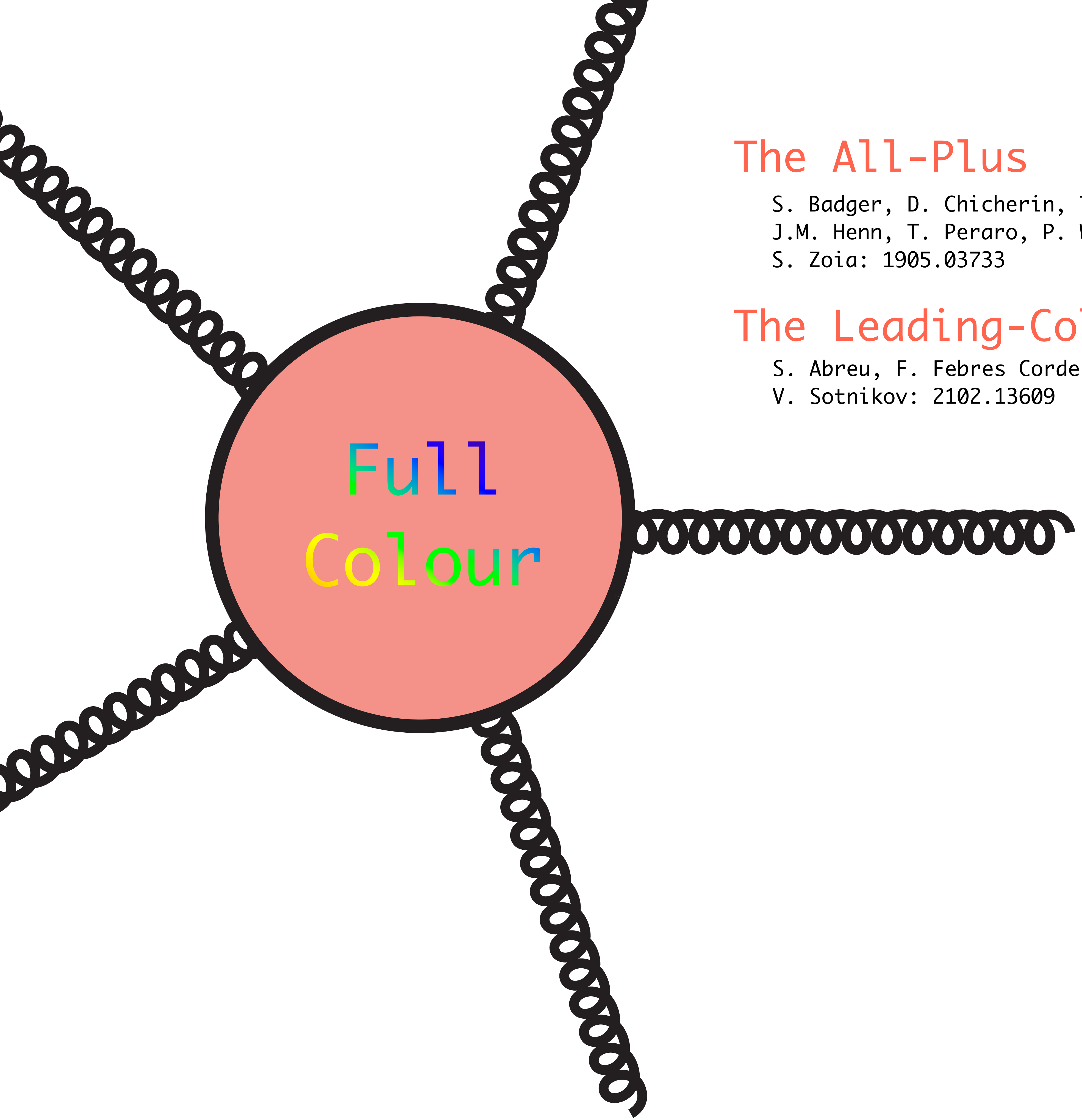


The All-Plus

S. Badger, D. Chicherin, T. Gehrmann, G. Heinrich, J.M. Henn, T. Peraro, P. Wasser, Y. Zhang, S. Zoia: 1905.03733

The Leading-Colour

S. Abreu, F. Febres Cordero, H. Ita, B. Page, V. Sotnikov: 2102.13609



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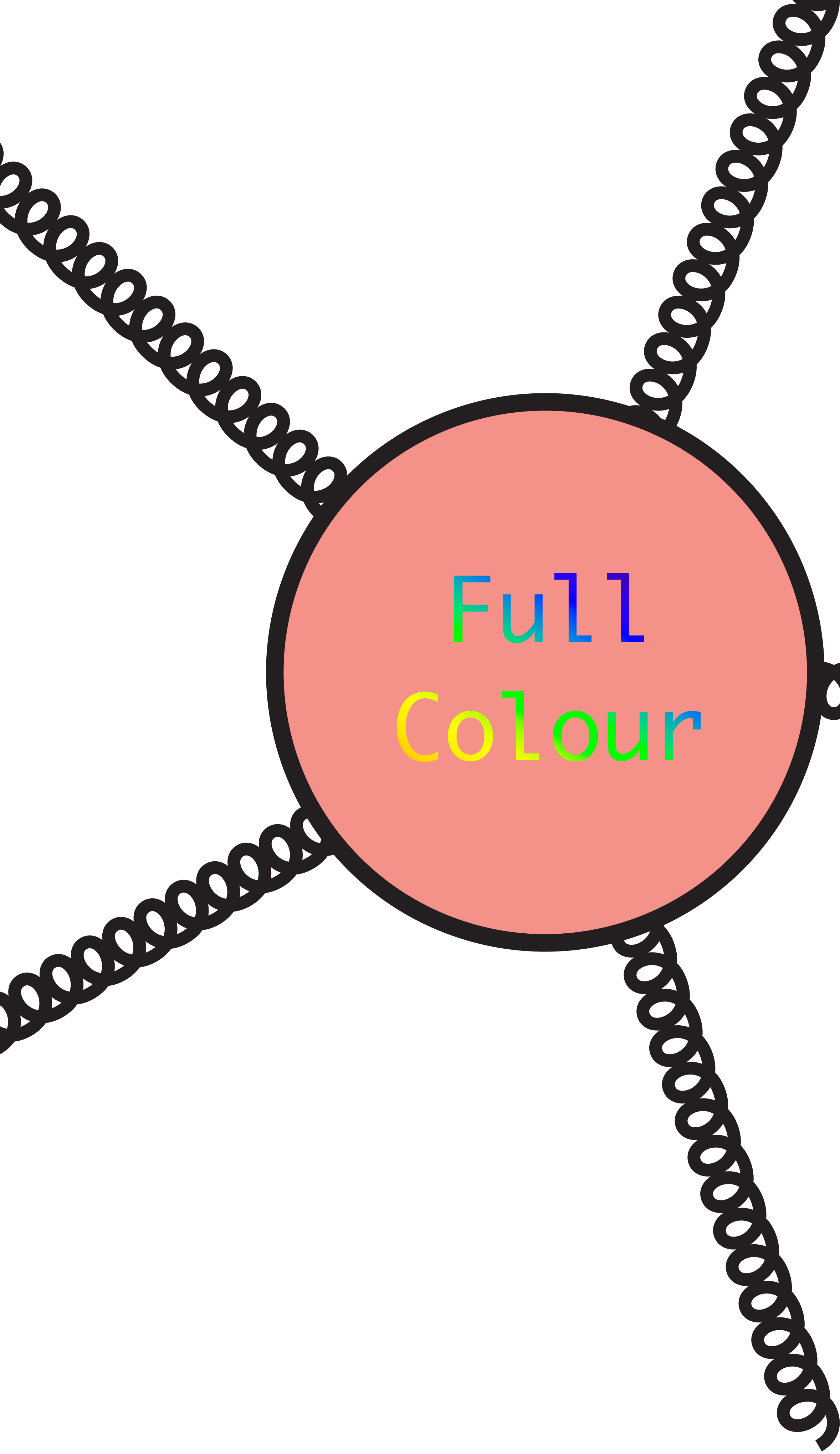

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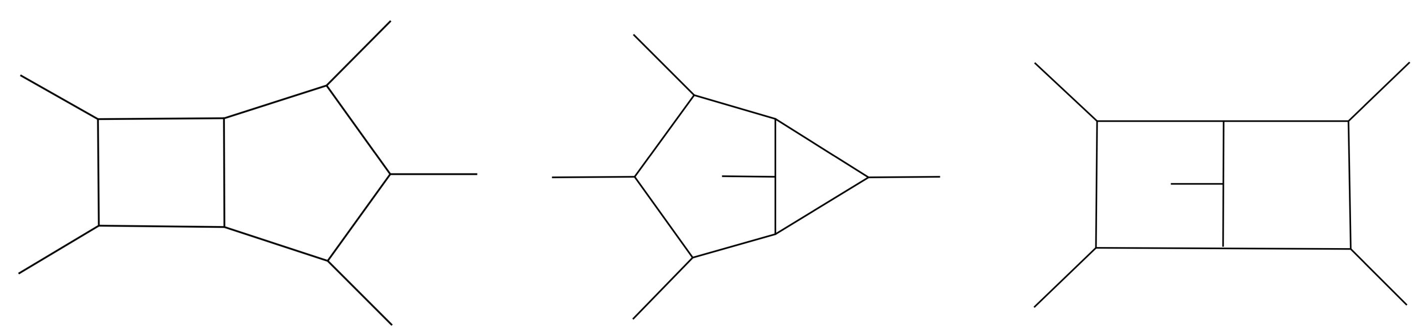
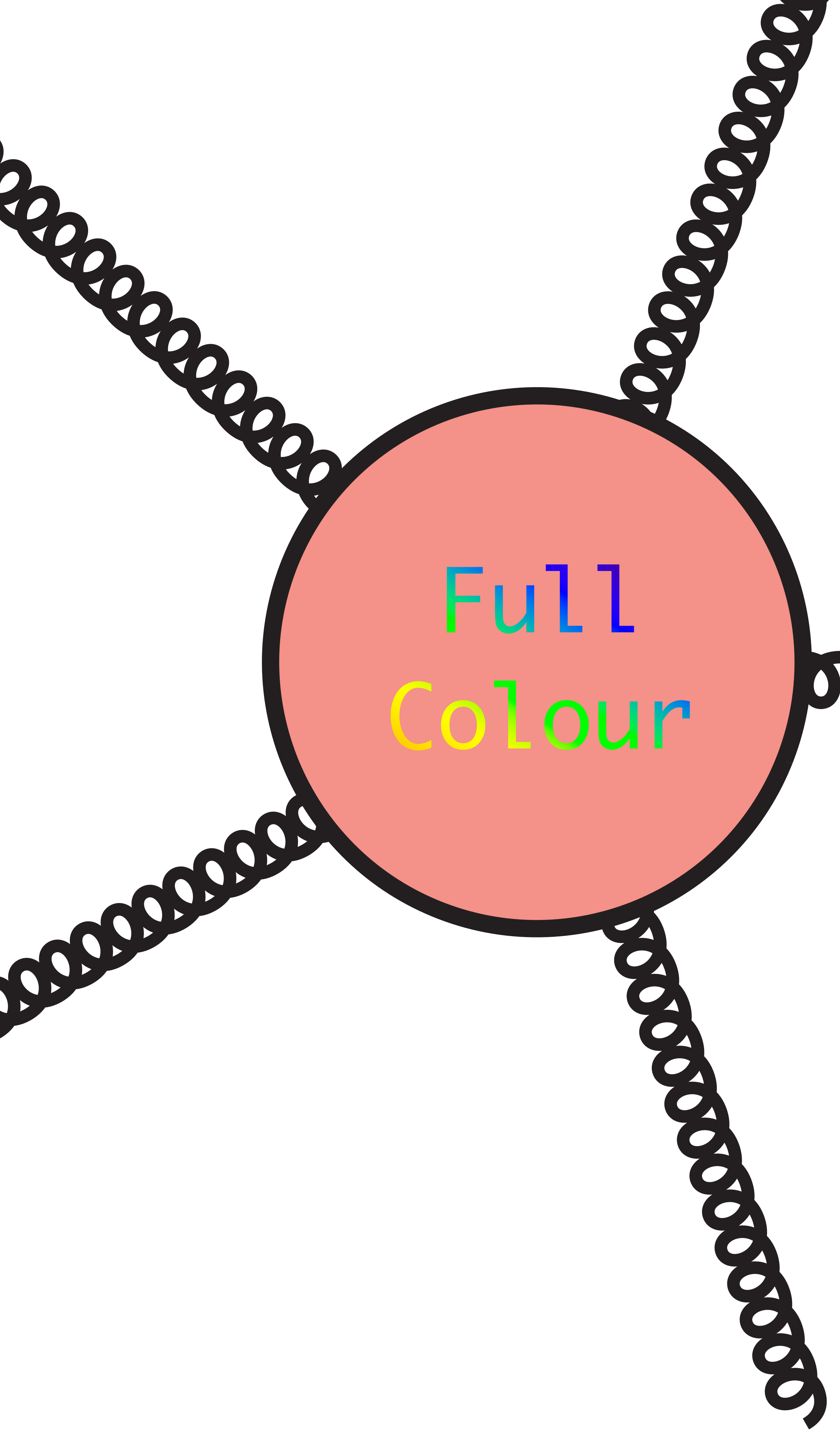
- All non-planar diagrams





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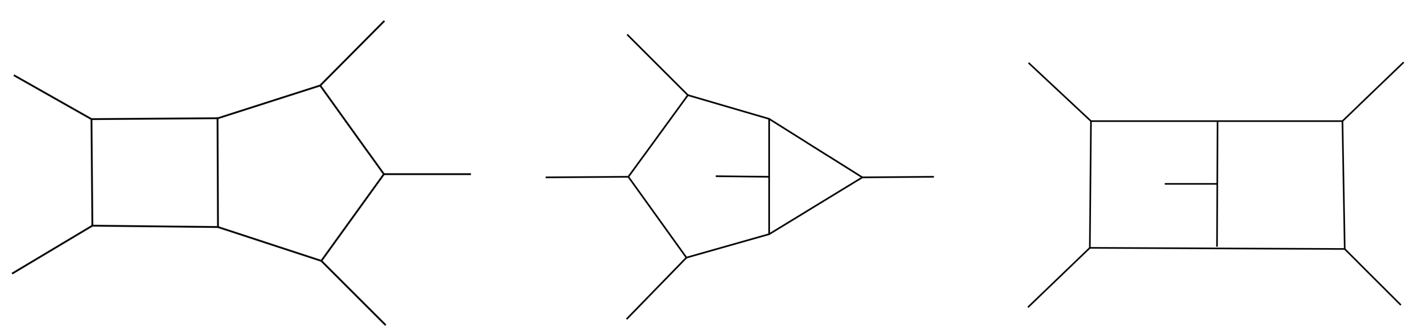
The Leading-Colour

S. Abreu, F. Febres Cordero, H. Ita, B. Page, V. Sotnikov: 2102.13609

- All non-planar diagrams
- Full dependence in N_c and n_f



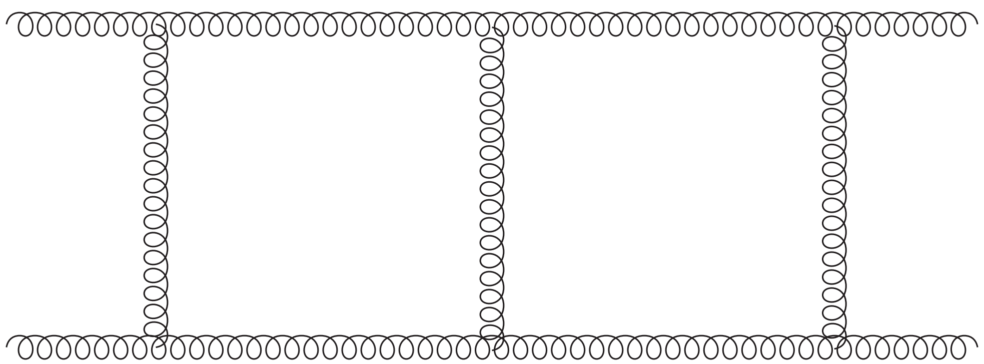
The Planar Limit

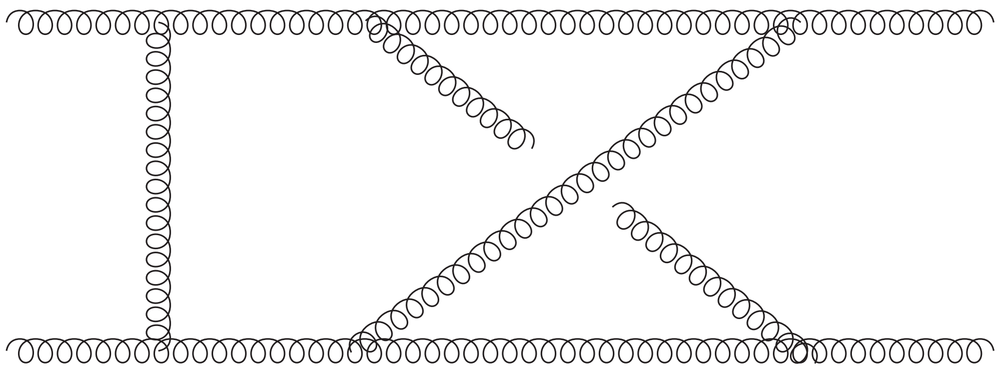
The Planar Limit

$$SU(N_c) \to U(N_c)$$

The Planar Limit

$$SU(N_c) \to U(N_c)$$





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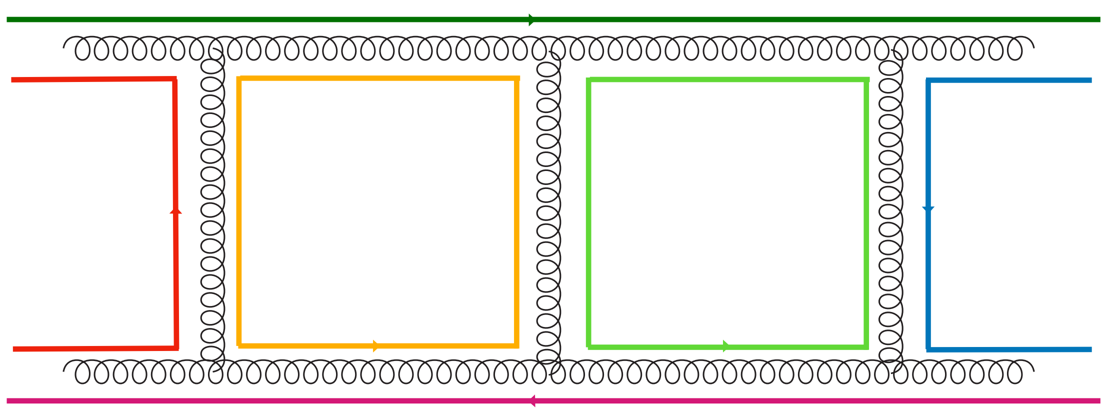
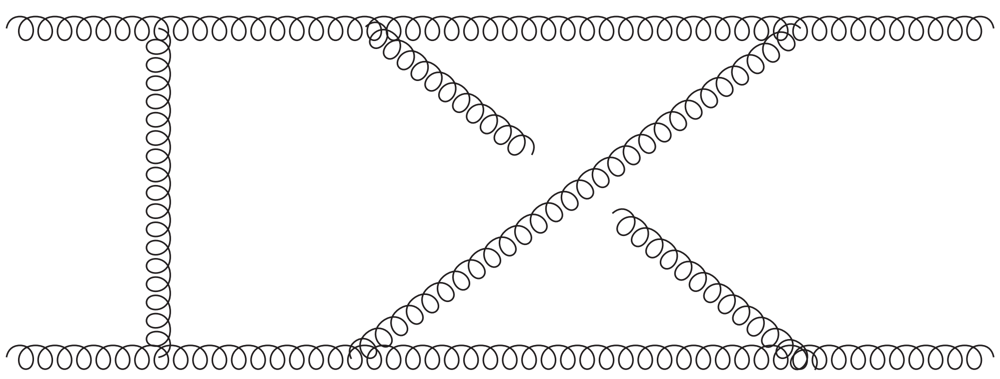
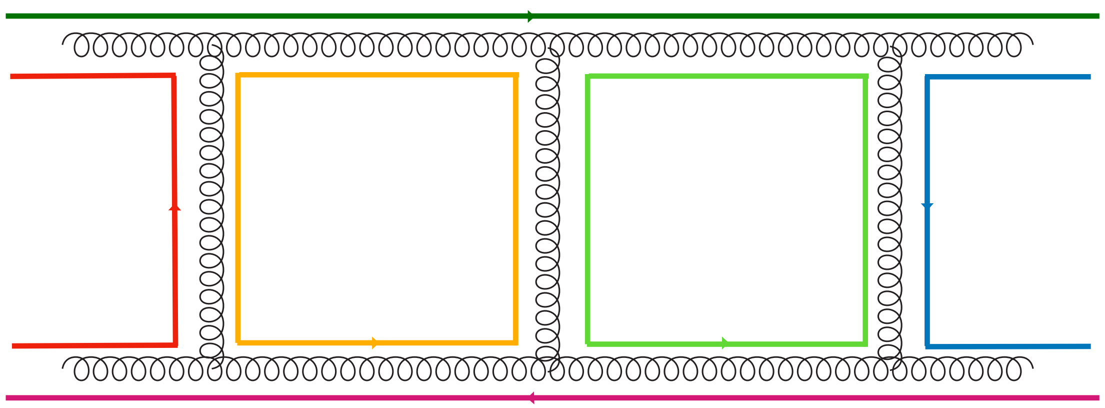
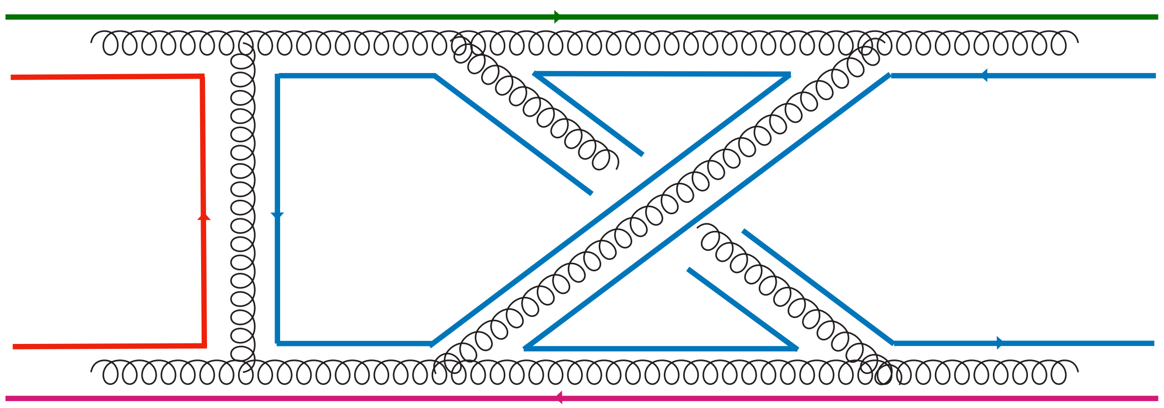




The Planar Limit

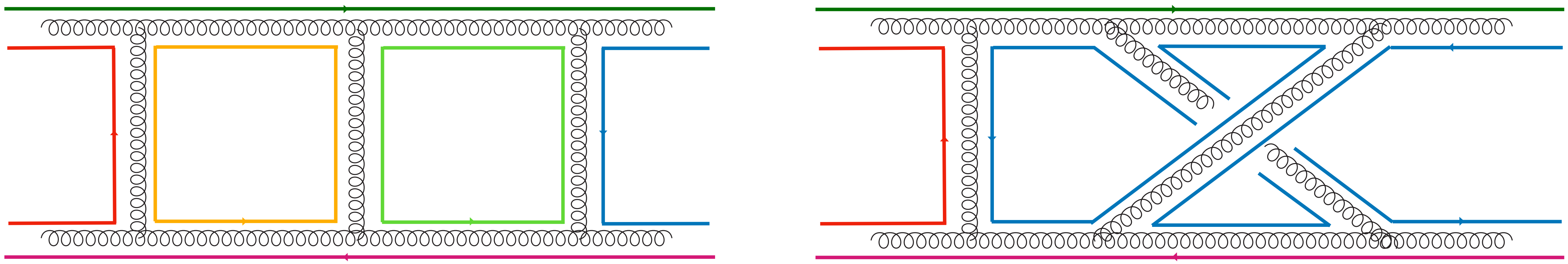
$$SU(N_c) \to U(N_c)$$





The Planar Limit

$$SU(N_c) \to U(N_c)$$

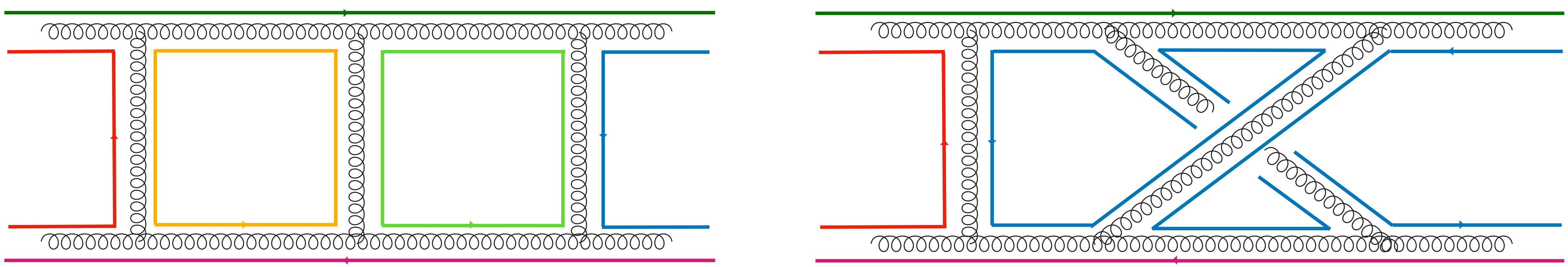


't Hooft coupling

$$\lambda = g^2 N_c$$

The Planar Limit

$$SU(N_c) \to U(N_c)$$



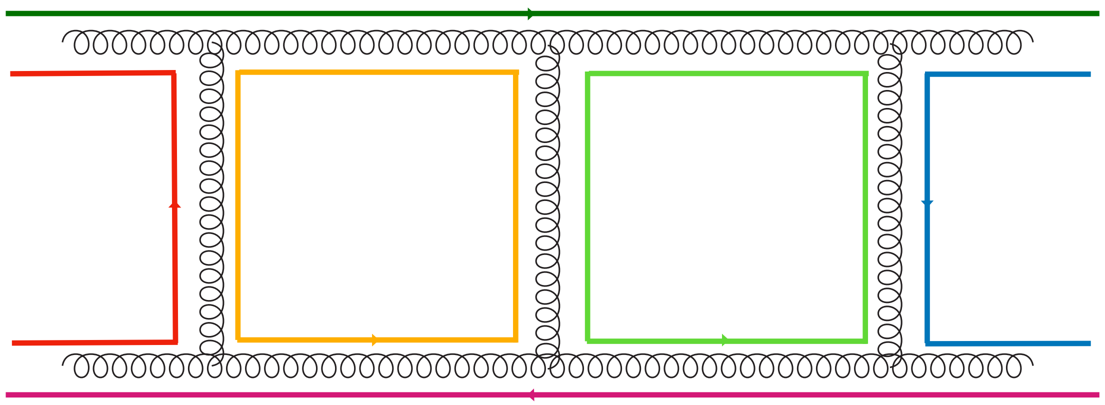
't Hoofft coupling

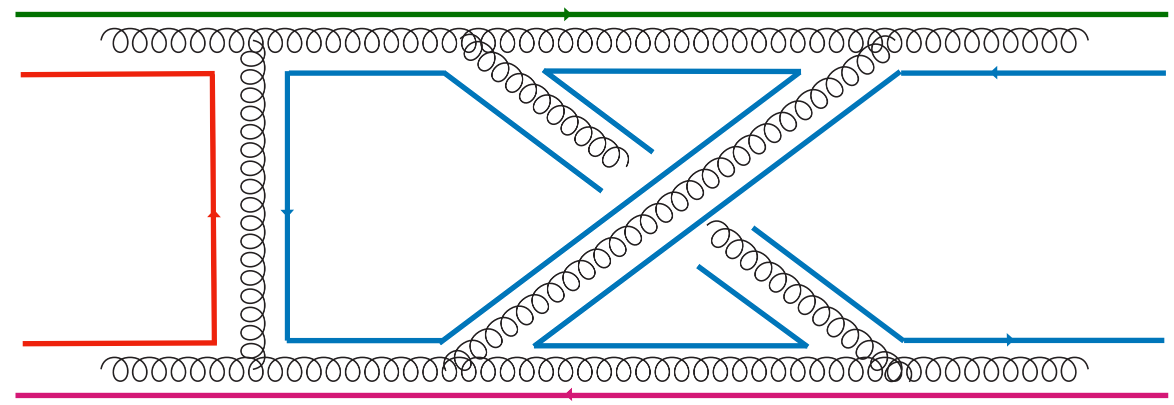
$$\lambda = g^2 N_c$$

$$\lambda^{\text{loops}} N_c^{\chi}$$

The Planar Limit

$$SU(N_c) \to U(N_c)$$





't Hooft coupling

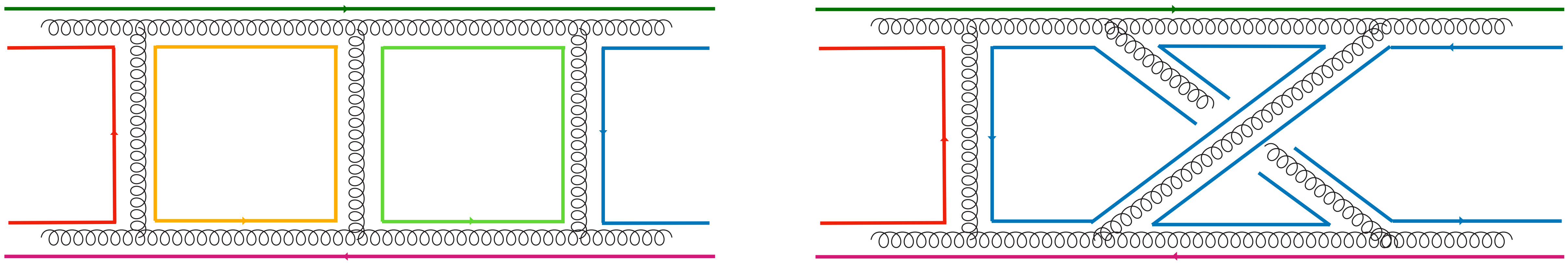
$$\lambda = g^2\, N_c$$

$$\lambda^{\text{loops}}\, N_c^{\chi}$$

Sphere	$\chi = 2$
Other Manifolds	$\chi = 2 - \text{holes}$

The Planar Limit

$$SU(N_c) \to U(N_c)$$



't Hooft coupling

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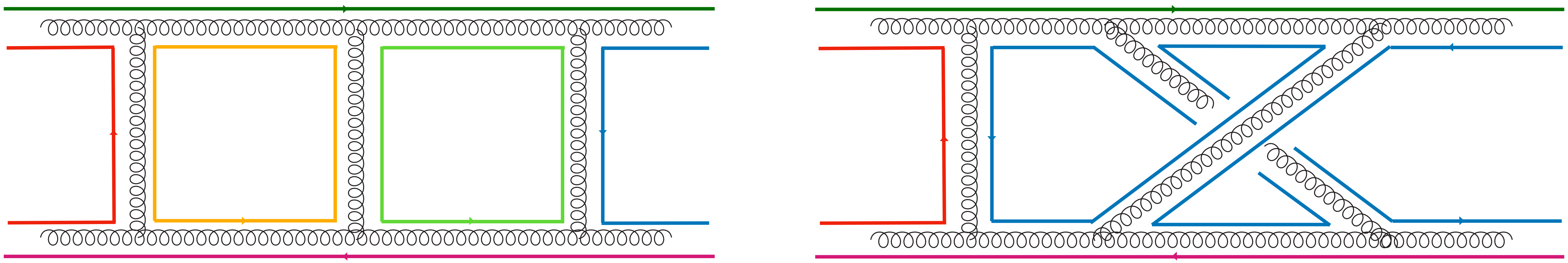
large N_c limit ↔ planar diagrams

The Planar Limit

$$SU(N_c) \to U(N_c)$$



't Hooft coupling

$$\lambda = g^2 N_c$$

$$\lambda^{\text{loops}} N_c^{\chi}$$

Sphere	$\chi = 2$
Other Manifolds	$\chi = 2 - \text{holes}$

large N_c limit ↔ planar diagrams

fewer diagrams, natural ordering, simpler kinematics

Full-Colour

Full-Colour

Colour Structures

$$\begin{array}{ll} 30 \to 74 & (ggggg) \\ 18 \to 51 & (q\bar{q}ggg) \\ 12 \to 24 & (q\bar{q}q'\bar{q}'g) \end{array}$$

Full-Colour

Colour Structures	$30 \to 74 \quad (ggggg)$ $18 \to 51 \quad (q\bar{q}ggg)$ $12 \to 24 \quad (q\bar{q}q'\bar{q}'g)$
Kinematic Poles	$5 \to 10$

Full-Colour

Colour Structures	$30 \rightarrow 74 \quad (ggggg)$ $18 \rightarrow 51 \quad (q\bar{q}ggg)$ $12 \rightarrow 24 \quad (q\bar{q}q'\bar{q}'g)$
Kinematic Poles	$5 \rightarrow 10$
Master Integrals	$61 \rightarrow 242$

Full-Colour

Colour Structures	$30 \to 74 \quad (ggggg)$ $18 \to 51 \quad (q\bar{q}ggg)$ $12 \to 24 \quad (q\bar{q}q'\bar{q}'g)$
Kinematic Poles	$5 \to 10$
Master Integrals	$61 \to 242$
Number of Terms	$\times 20$

Full-Colour

Colour Structures	$30 \to 74 \quad (ggggg)$ $18 \to 51 \quad (q\bar{q}ggg)$ $12 \to 24 \quad (q\bar{q}q'\bar{q}'g)$
Kinematic Poles	$5 \to 10$
Master Integrals	$61 \to 242$
Number of Terms	$\times 20$

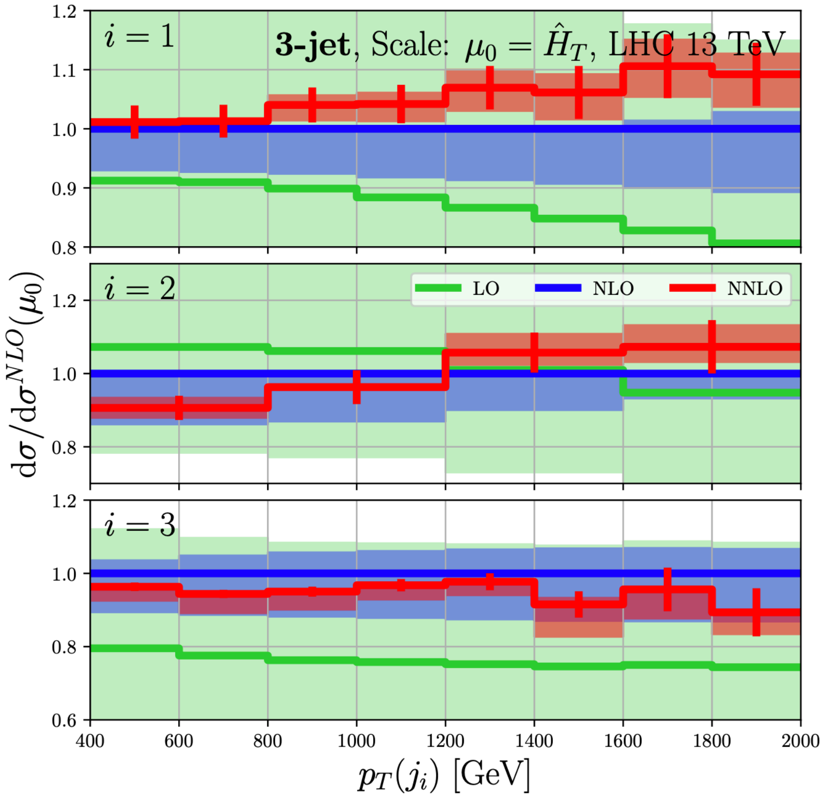
<u>Not reflected in the final result!</u>

Phenomenological Input

Phenomenological Input

2106.05331

3-jet cross section @NNLO

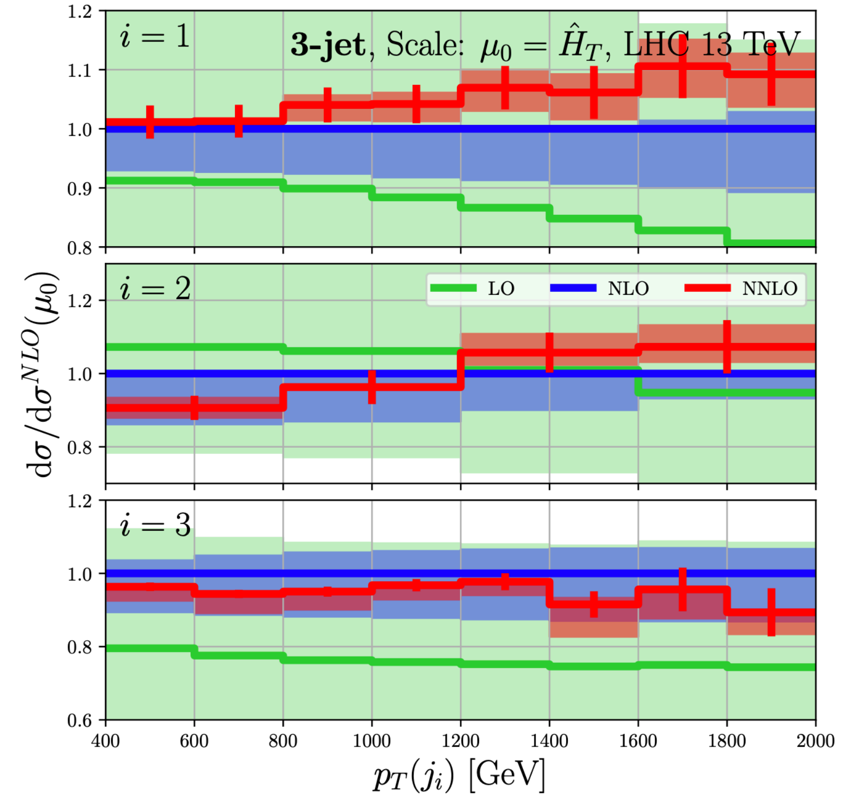


Phenomenological Input

2106.05331

3-jet cross section @NNLO

towards 2-jet cross section @N^{3}LO

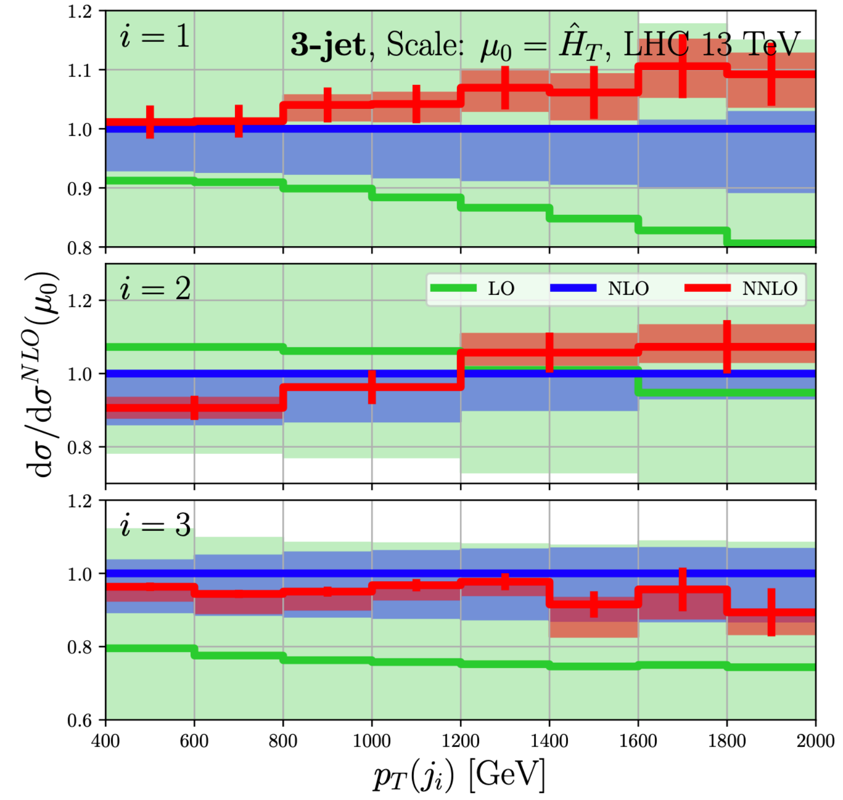


Phenomenological Input

2106.05331

3-jet cross section @NNLO

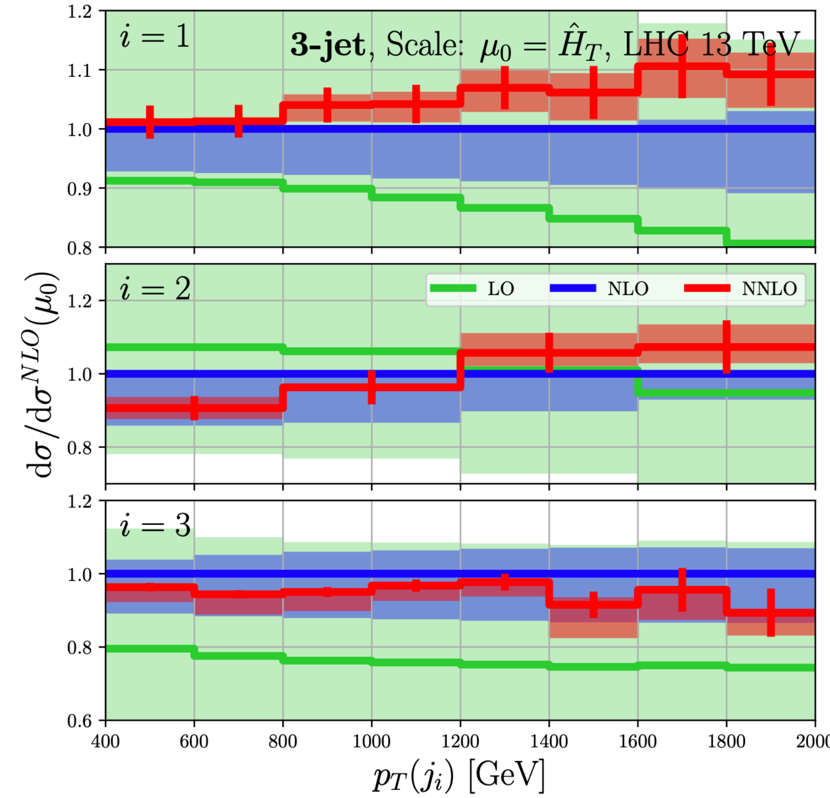
towards 2-jet cross section @N^3LO



Soft-Collinear Limit

Phenomenological Input

2106.05331



3-jet cross section @NNLO

towards 2-jet cross section @N^{3}LO

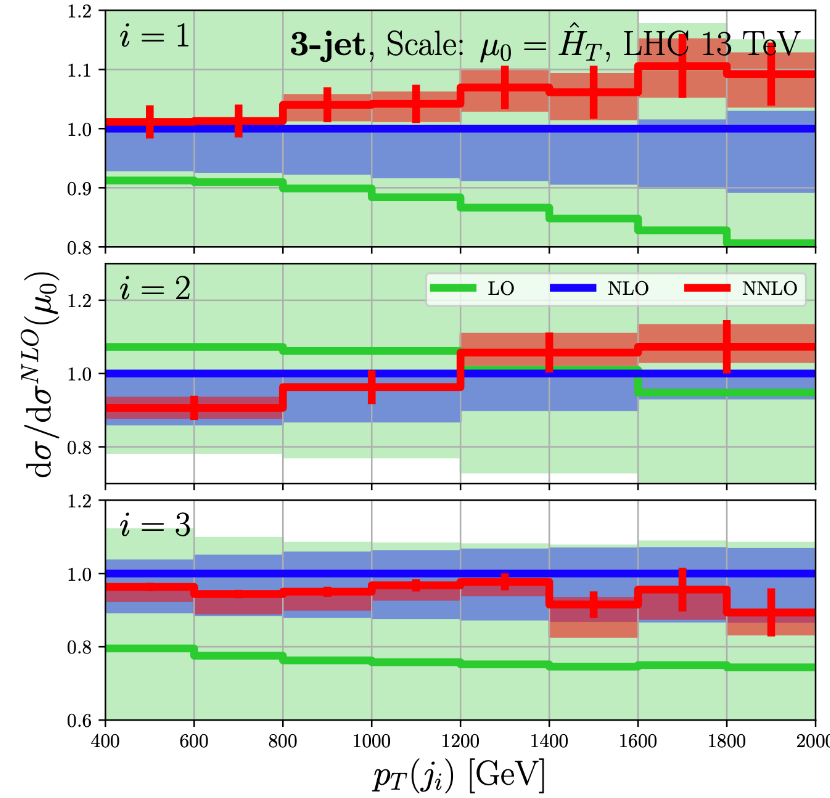
Soft-Collinear Limit

Phenomenological Input

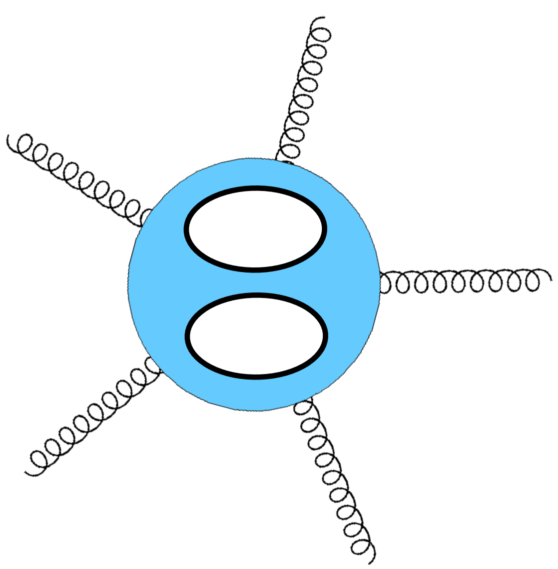
2106.05331

3-jet cross section @NNLO

towards 2-jet cross section @N^{3}LO

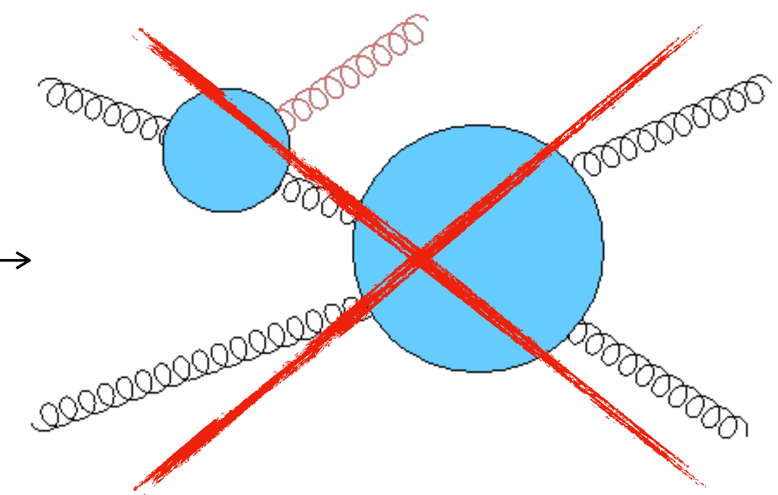


Soft-Collinear Limit



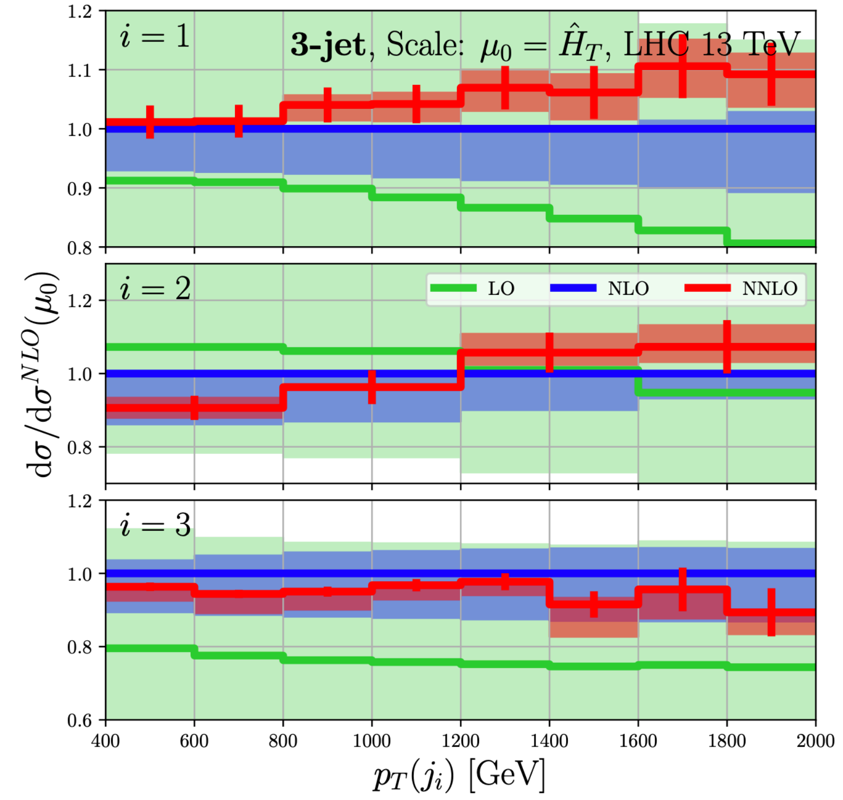
Catani, de Florian, Rodrigo: 1112.4405

Dixon, Herrmann, Yan, Zhu: 1912.09370



Phenomenological Input

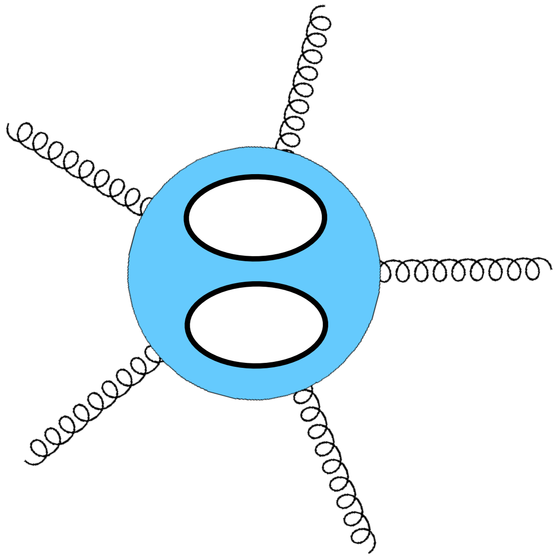
2106.05331



3-jet cross section @NNLO

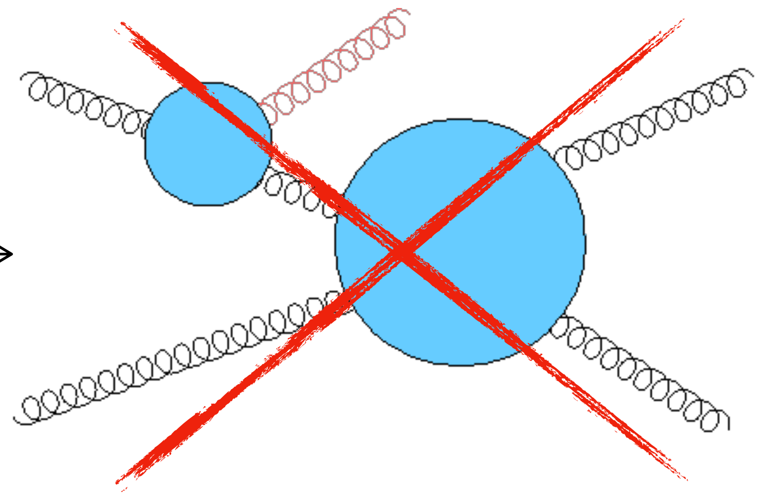
towards 2-jet cross section @N^{3}LO

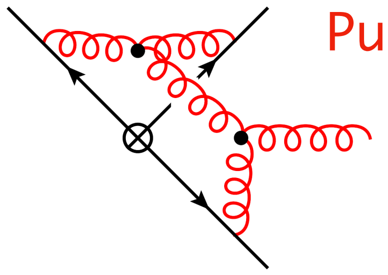
Soft-Collinear Limit



Catani, de Florian, Rodrigo: 1112.4405

Dixon, Herrmann, Yan, Zhu: 1912.09370





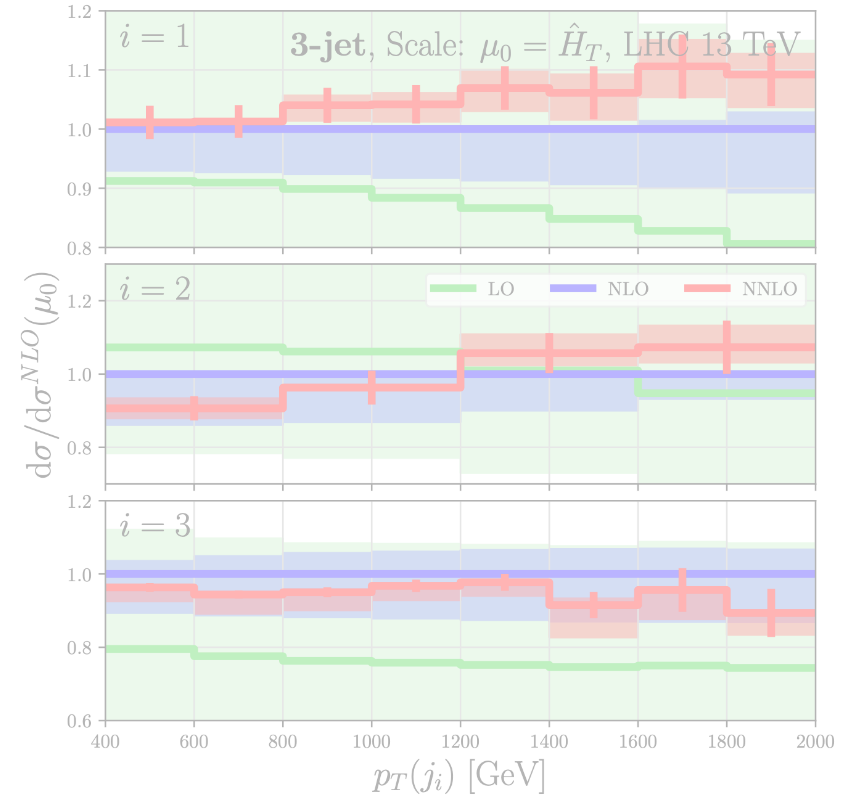
Purely non-planar!

Phenomenological Input

2106.05331

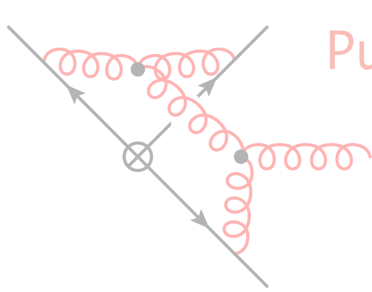
3-jet cross section @NNLO

towards 2-jet cross section @N³LO



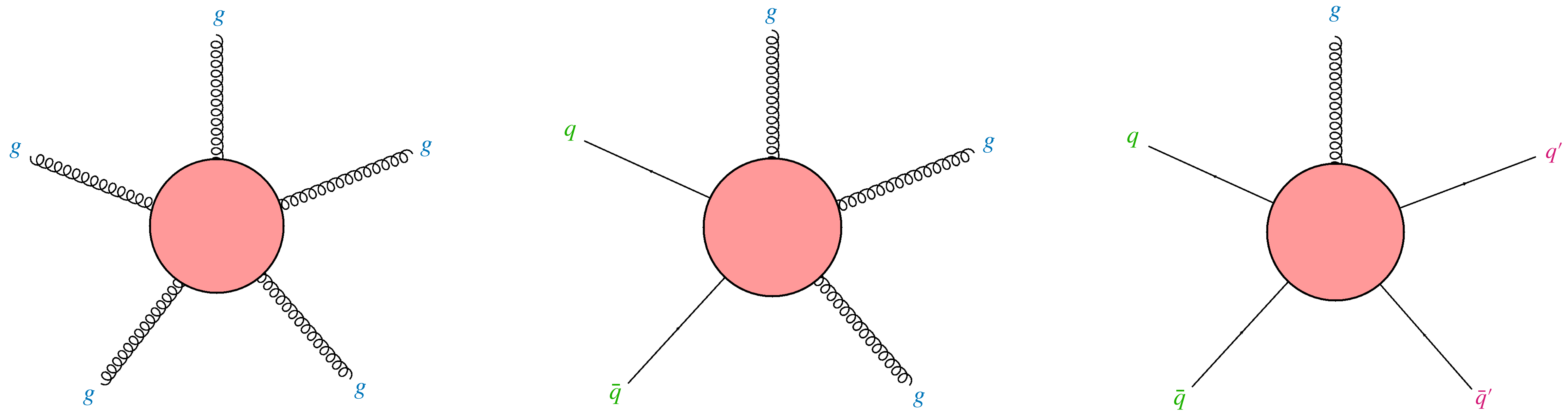
Soft-Collinear Limit

but how?

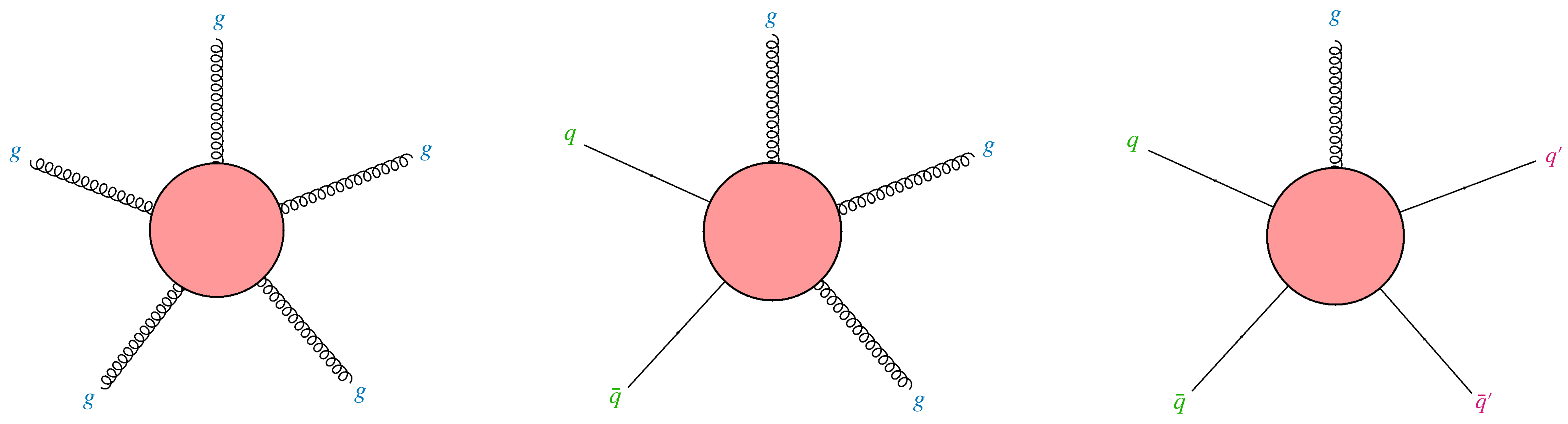


Divide et Impera

Primitive Amplitudes

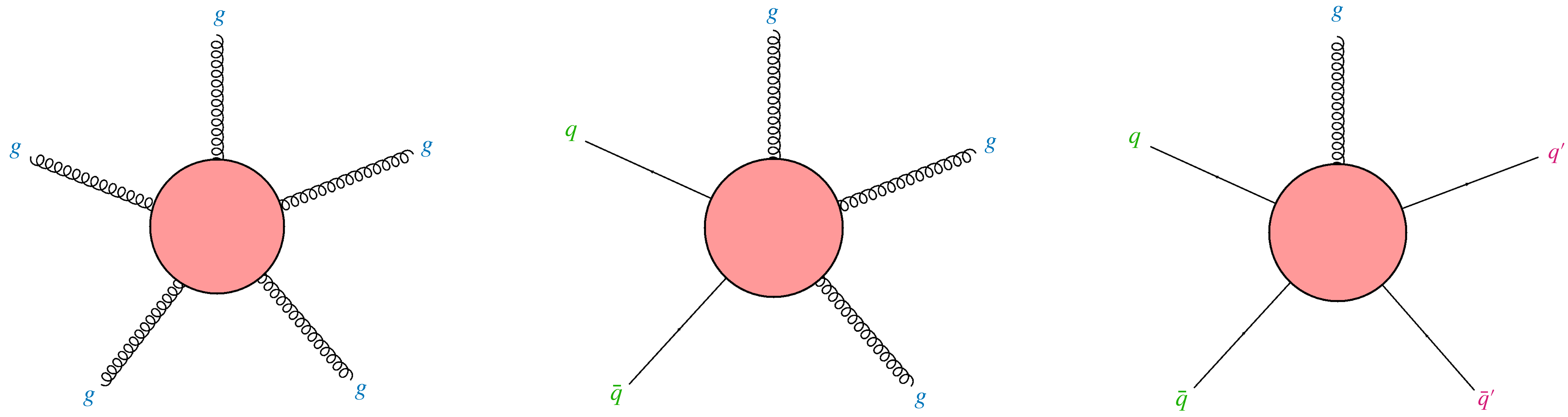


Primitive Amplitudes



crossing symmetry

Primitive Amplitudes



crossing symmetry

$$gg \to ggg,$$

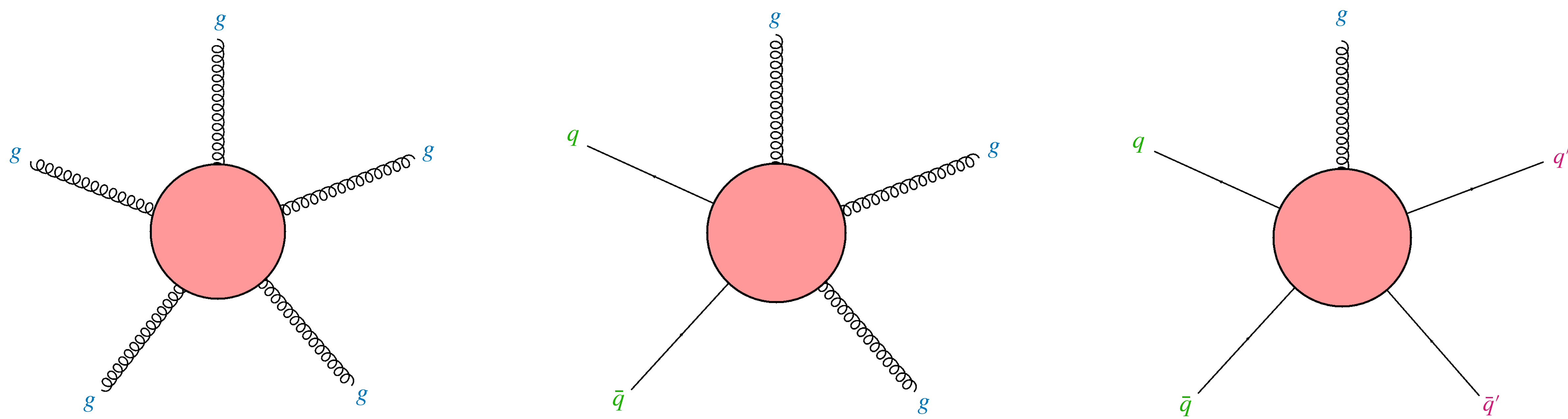
$$q\bar{q} \to ggg, \quad qg \to qgg, \quad \bar{q}g \to \bar{q}gg, \quad gg \to q\bar{q}g,$$

$$q\bar{q} \to q'\bar{q}'g, \quad q\bar{q}' \to \bar{q}q'g, \quad qq' \to qq'g, \quad \bar{q}\bar{q}' \to \bar{q}\bar{q}'g,$$

$$qg \to qq'\bar{q}', \quad \bar{q}g \to \bar{q}q'\bar{q}'.$$

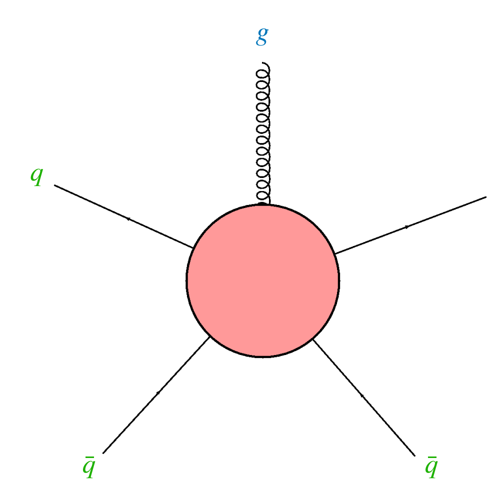
all colours, all helicities

Primitive Amplitudes

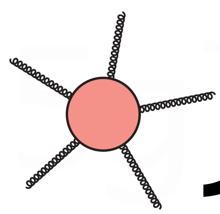


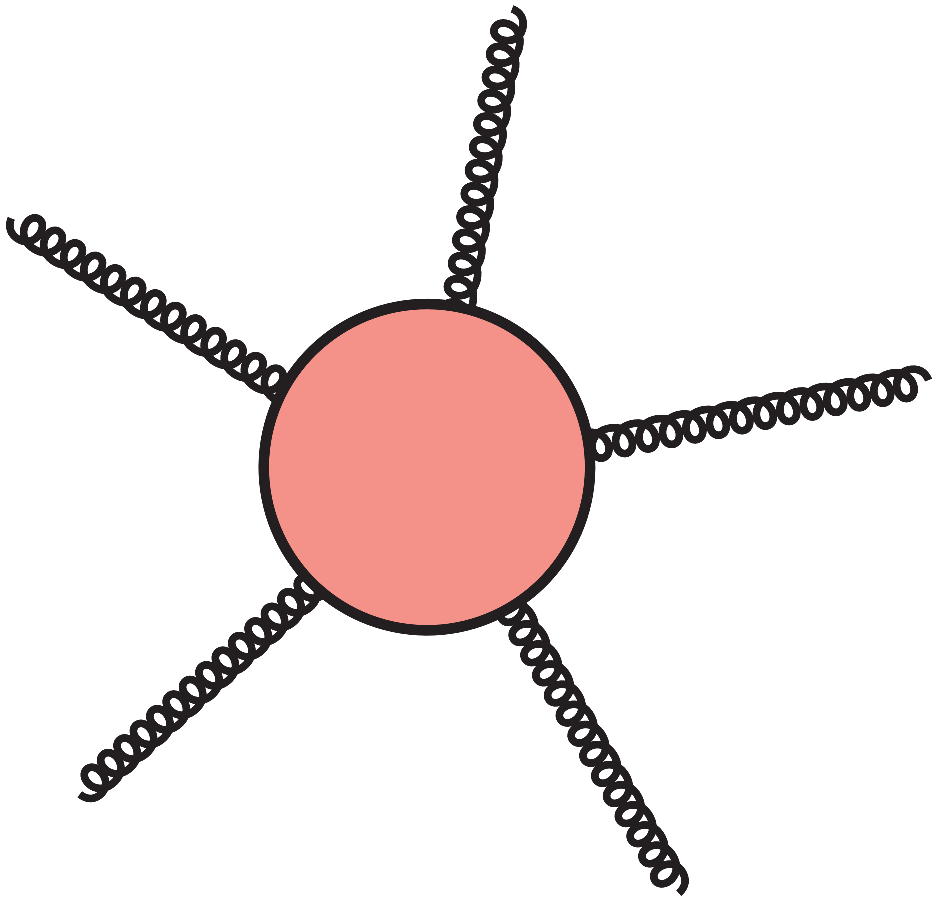
crossing symmetry

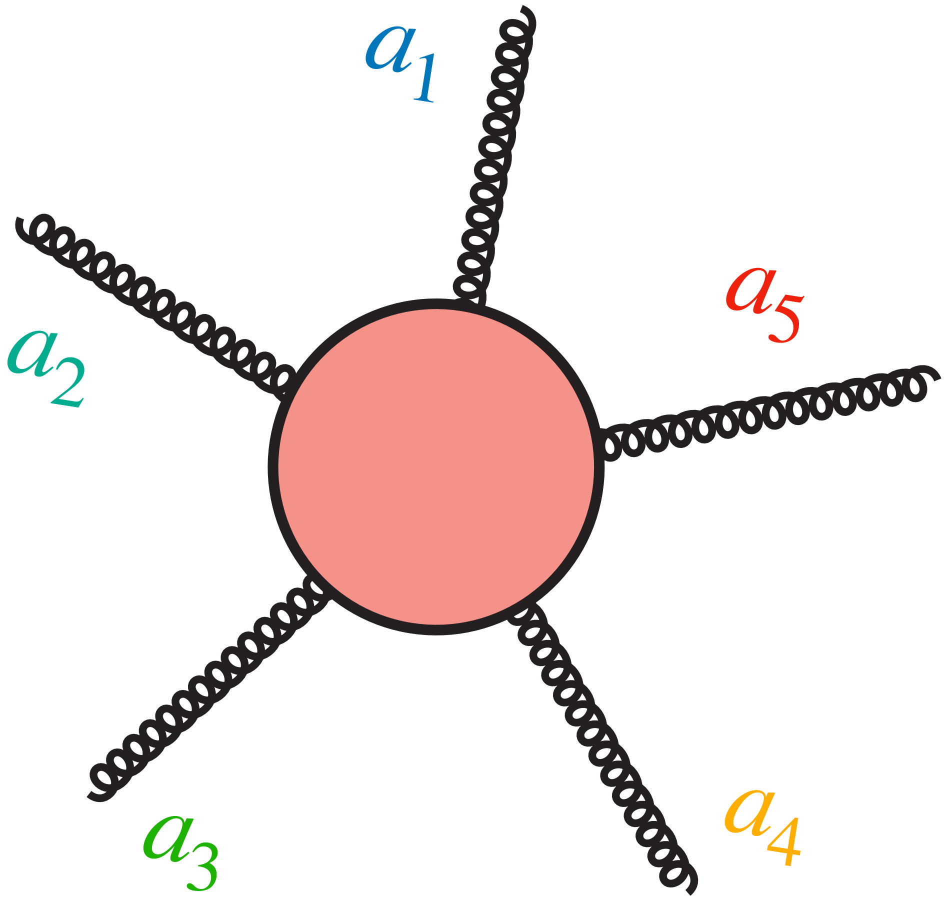
$$gg \to ggg,$$
$$q\bar{q} \to ggg, \quad qg \to qgg, \quad \bar{q}g \to \bar{q}gg, \quad gg \to q\bar{q}g,$$
$$q\bar{q} \to q'\bar{q}'g, \quad q\bar{q}' \to \bar{q}q'g, \quad qq' \to qq'g, \quad \bar{q}\bar{q}' \to \bar{q}\bar{q}'g,$$
$$qg \to qq'\bar{q}', \quad \bar{q}g \to \bar{q}q'\bar{q}'.$$

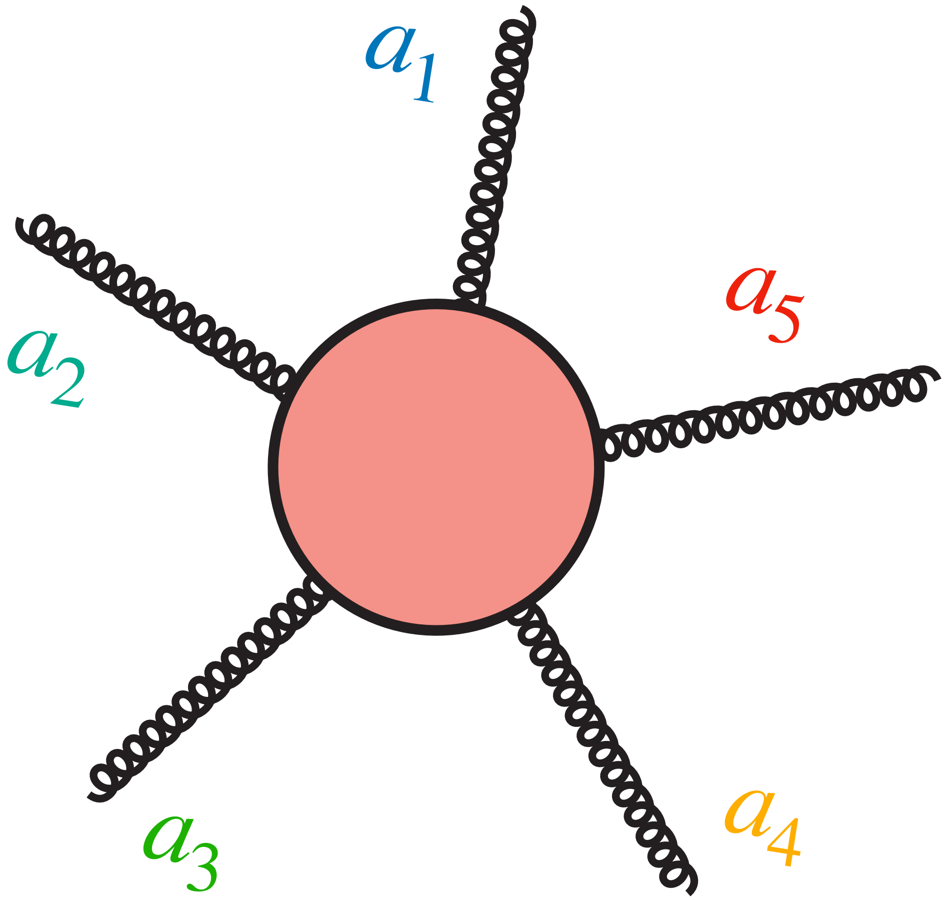


all colours, all helicities

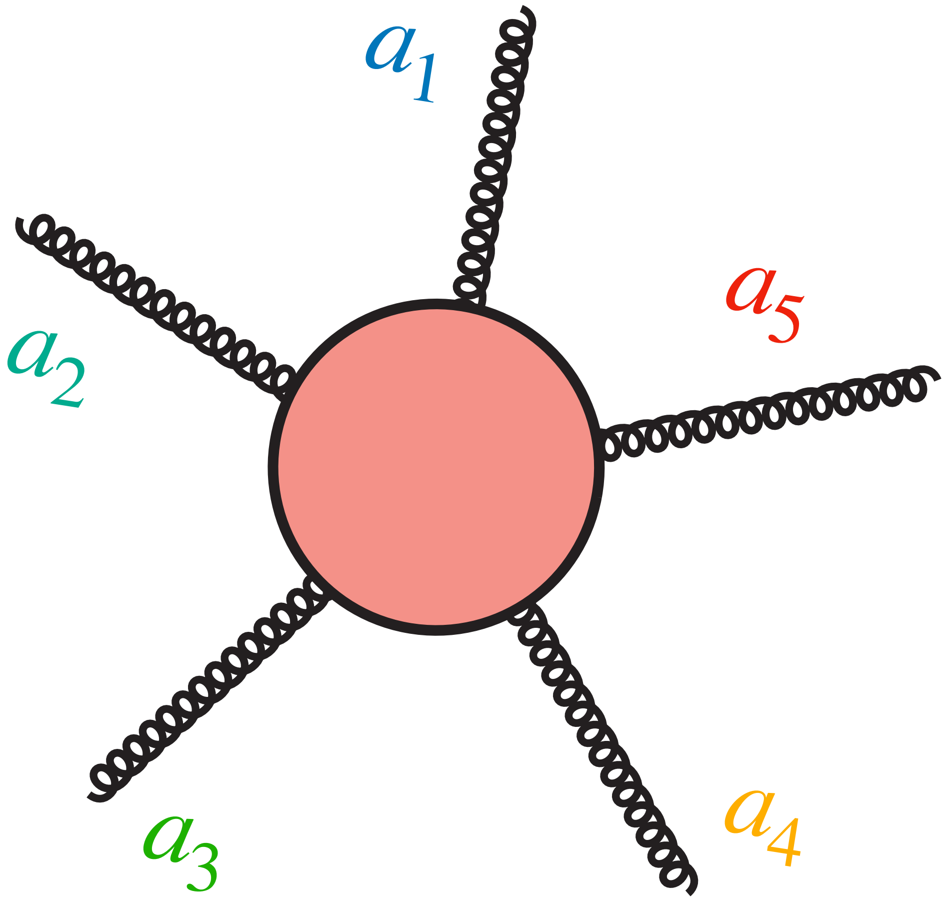
The () Computation




a_1
a_2
a_3
a_4
a_5

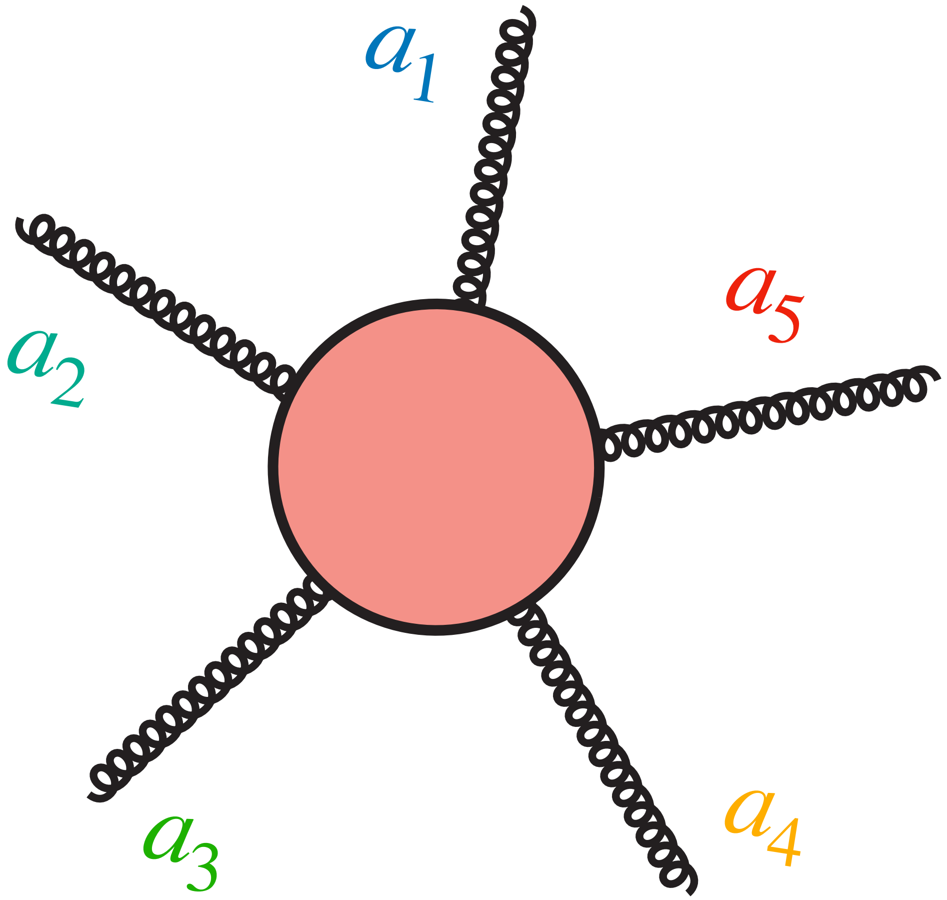


$$A^{a_1 a_2 \dots a_5} = \sum_{c=1}^{22} A_c \, \mathscr{C}_c^{a_1 a_2 \dots a_5}$$



$$A^{a_1 a_2 \ldots a_5} = \sum_{c=1}^{22} A_c \, \mathscr{C}_c^{a_1 a_2 \ldots a_5}$$

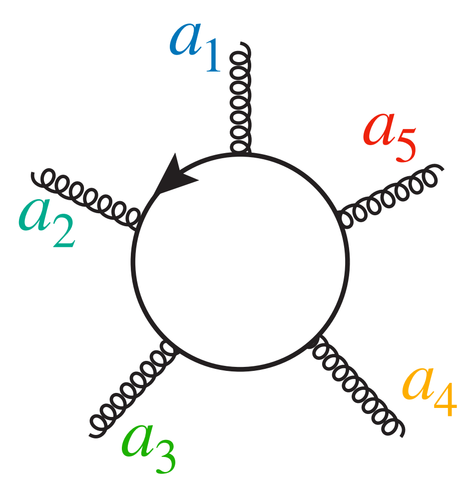
Partial
Amplitudes

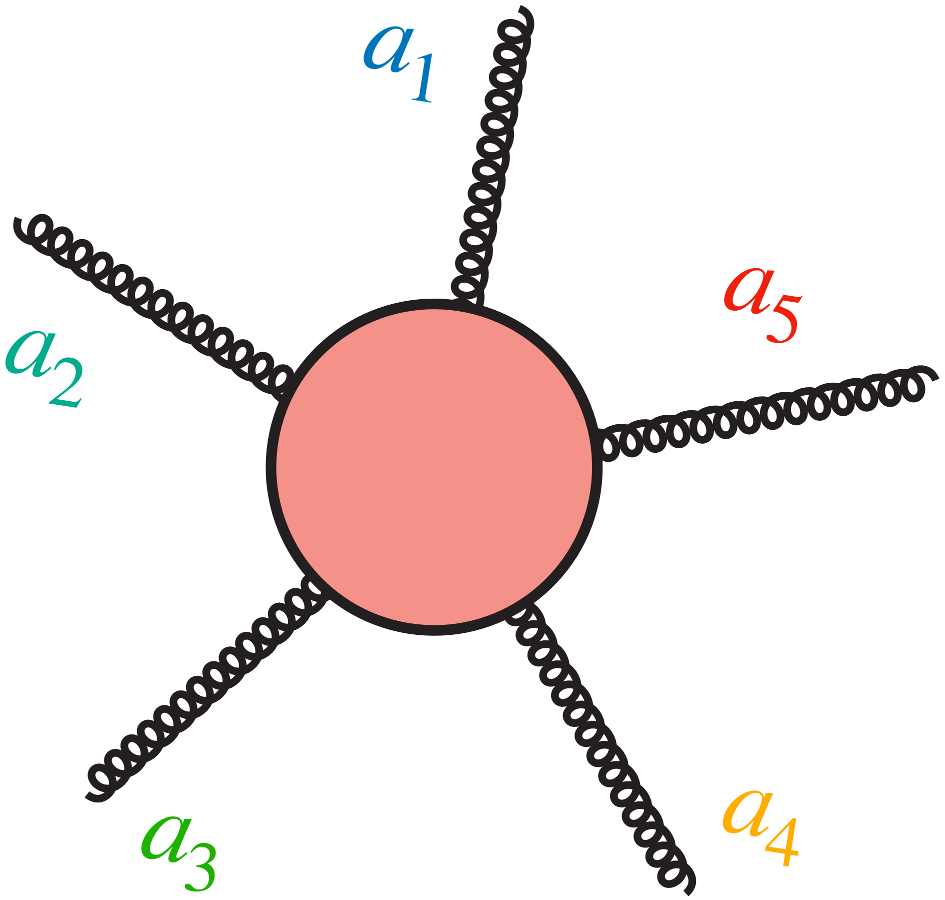


$$A^{a_1 a_2 \dots a_5} = \sum_{c=1}^{22} A_c \, \mathcal{C}_c^{a_1 a_2 \dots a_5}$$

Partial Amplitudes

$$A^{a_1 a_2 \dots a_5} = A_1$$

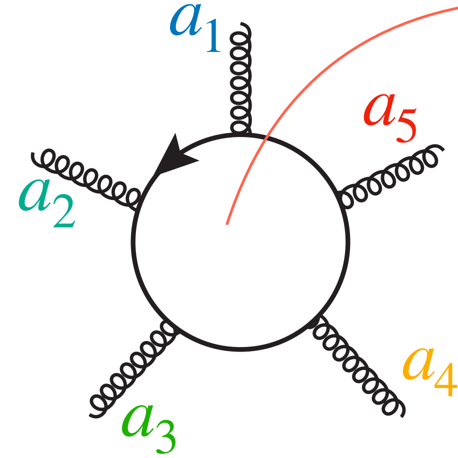




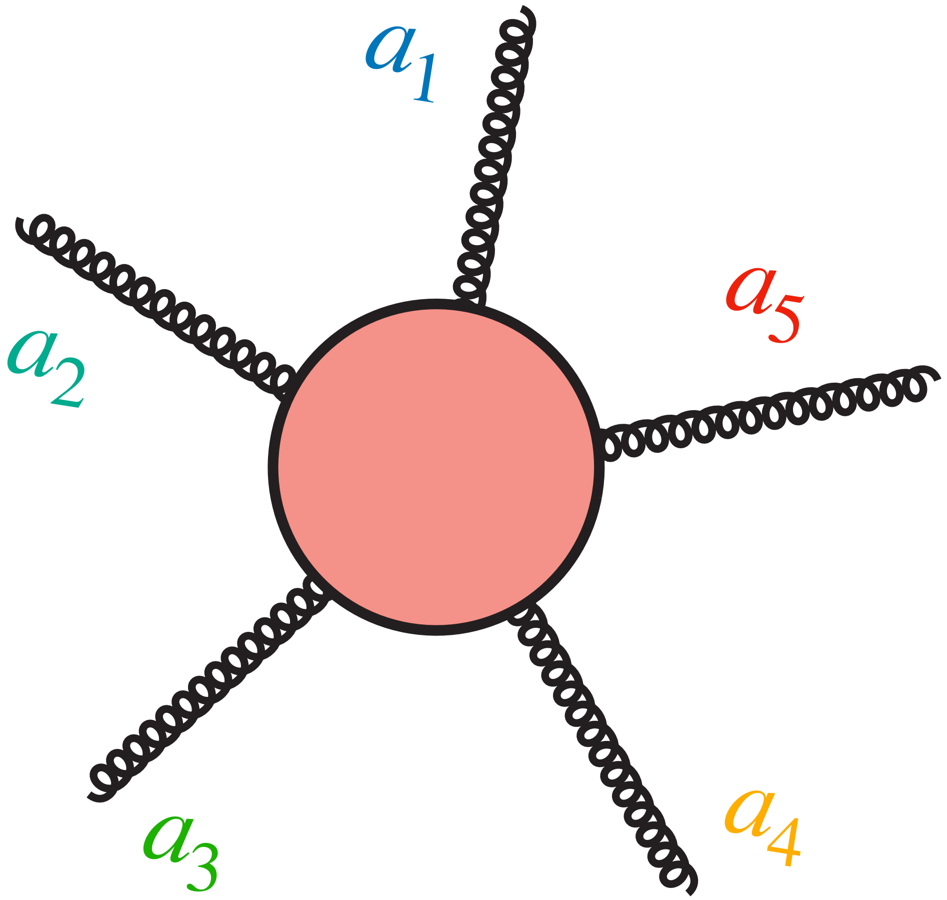
$$A^{a_1 a_2 \ldots a_5} = \sum_{c=1}^{22} A_c \, \mathscr{C}_c^{a_1 a_2 \ldots a_5}$$

Partial Amplitudes

$$A^{a_1 a_2 \ldots a_5} = \quad A_1$$



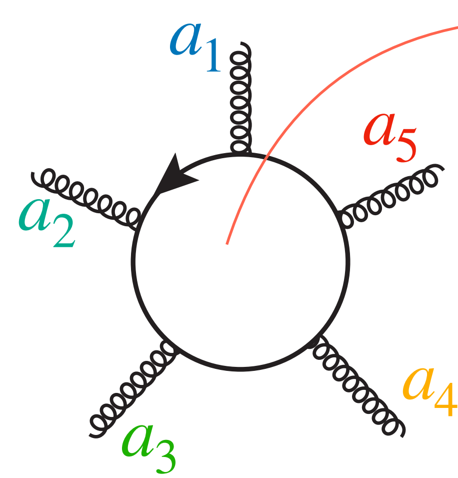
$Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4}\mathbf{T}^{a_5})$

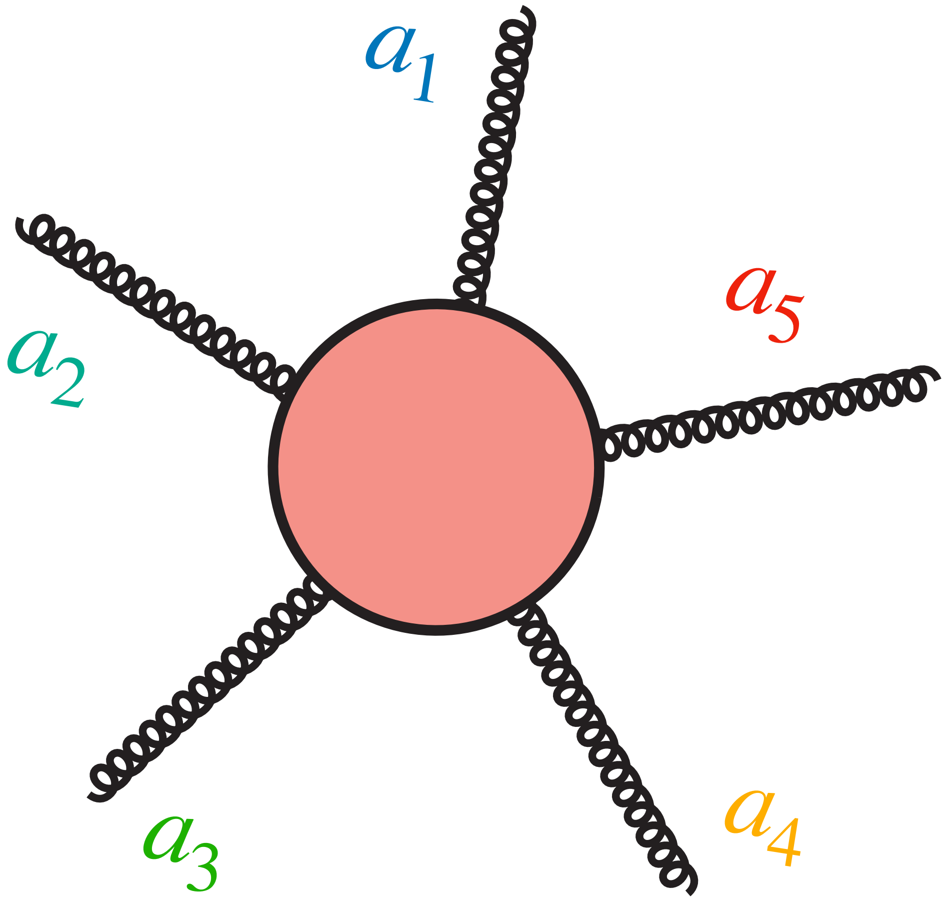


$$A^{a_1 a_2 \ldots a_5} = \sum_{c=1}^{22} A_c \, \mathscr{C}_c^{a_1 a_2 \ldots a_5}$$

Partial Amplitudes

$Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4}\mathbf{T}^{a_5})$

$A^{a_1 a_2 \ldots a_5} = \quad A_1$  + 11 permutations …

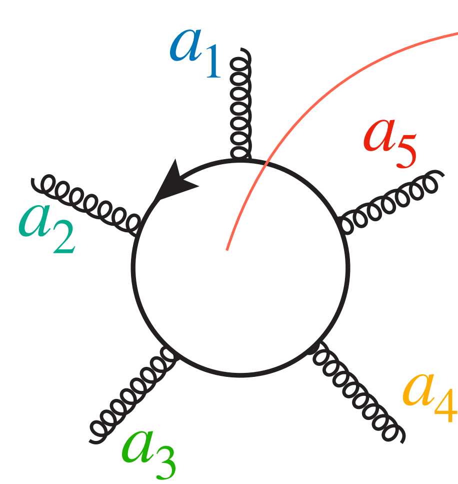


$$A^{a_1 a_2 \ldots a_5} = \sum_{c=1}^{22} A_c \, \mathscr{C}_c^{a_1 a_2 \ldots a_5}$$

Partial Amplitudes

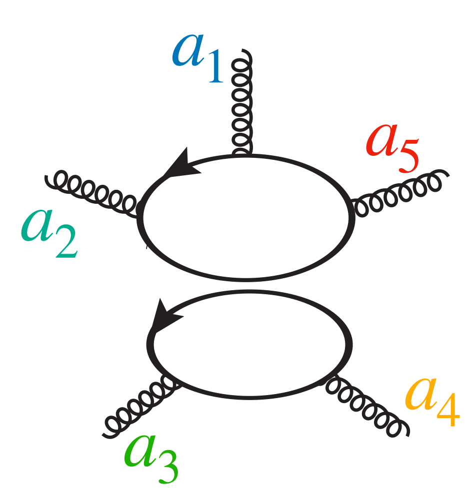
$Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4}\mathbf{T}^{a_5})$

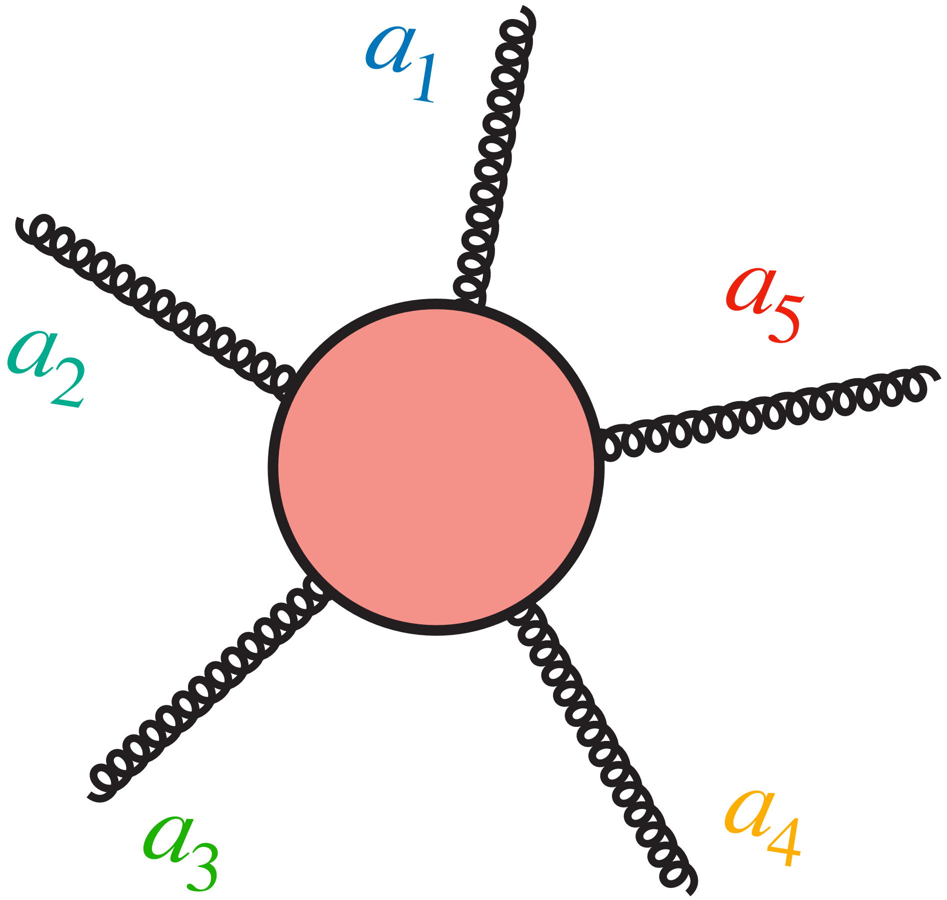
$$A^{a_1 a_2 \ldots a_5} = \quad A_1$$



+ 11 permutations …

$$+ \quad A_2$$



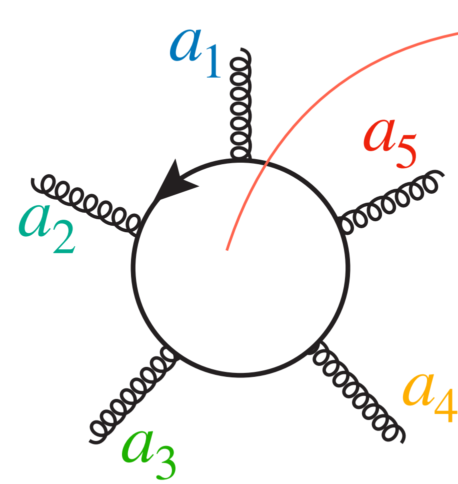


$$A^{a_1 a_2 \ldots a_5} = \sum_{c=1}^{22} A_c \, \mathscr{C}_c^{a_1 a_2 \ldots a_5}$$

Partial Amplitudes

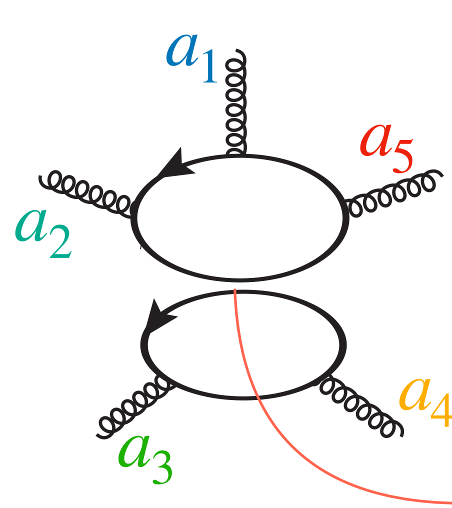
$Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4}\mathbf{T}^{a_5})$

$A^{a_1 a_2 \ldots a_5} = \quad A_1$

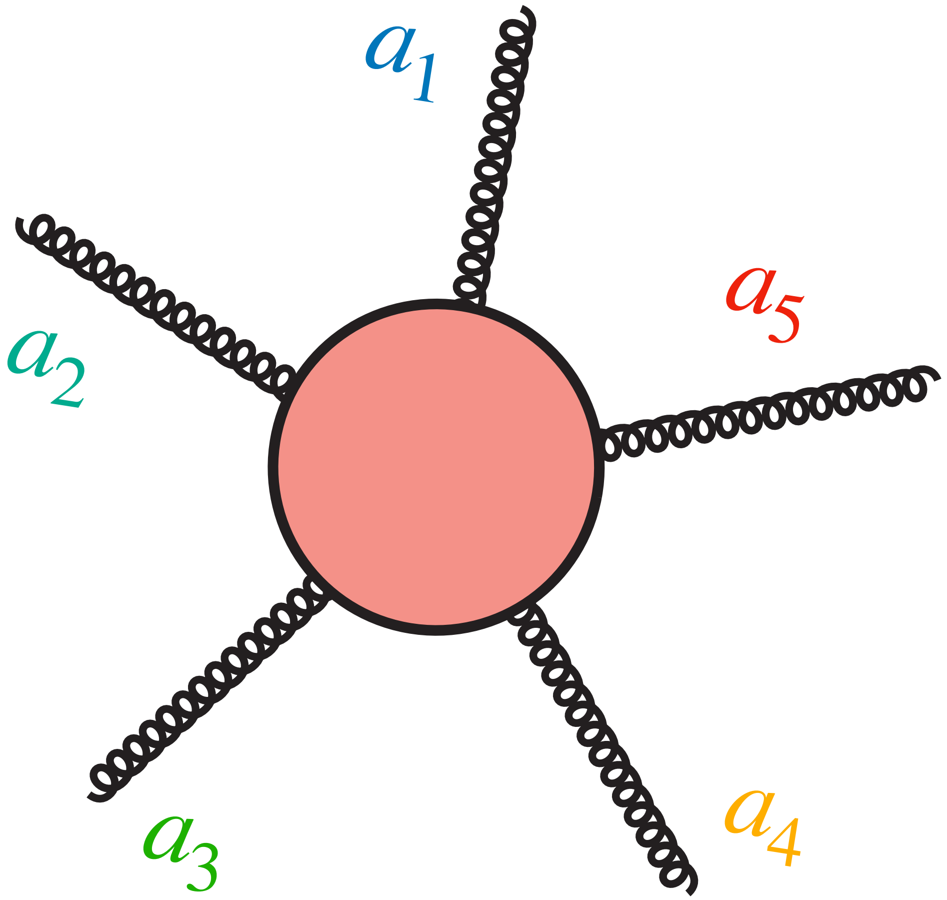


+ 11 permutations …

$+ \quad A_2$



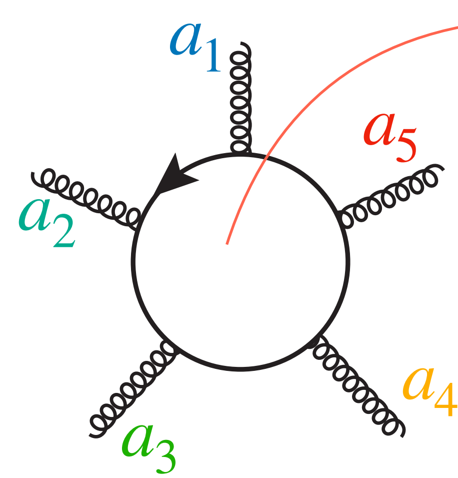
$Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_5}) \; Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_4})$



$$A^{a_1 a_2 \ldots a_5} = \sum_{c=1}^{22} A_c \, \mathscr{C}_c^{a_1 a_2 \ldots a_5}$$

Partial Amplitudes

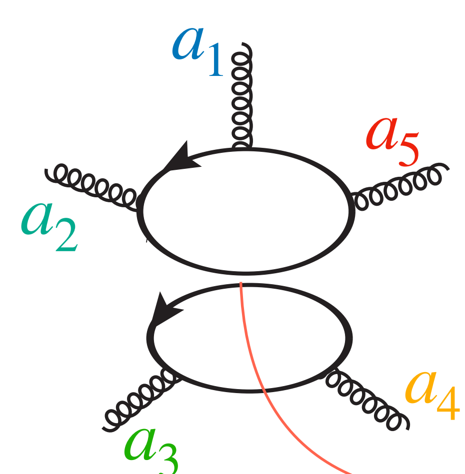
$$A^{a_1 a_2 \ldots a_5} = \quad A_1$$



$Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_3}\mathbf{T}^{a_4}\mathbf{T}^{a_5})$

+ 11 permutations …

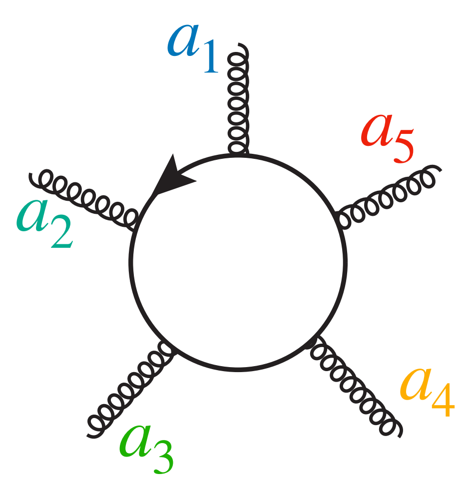
$$+ \quad A_2$$



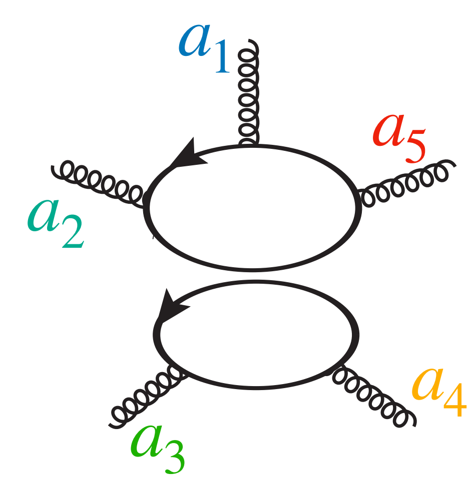
+ 9 permutations …

$Tr(\mathbf{T}^{a_1}\mathbf{T}^{a_2}\mathbf{T}^{a_5})\ Tr(\mathbf{T}^{a_3}\mathbf{T}^{a_4})$

$$A^{a_1 a_2 \ldots a_5} = A_1$$

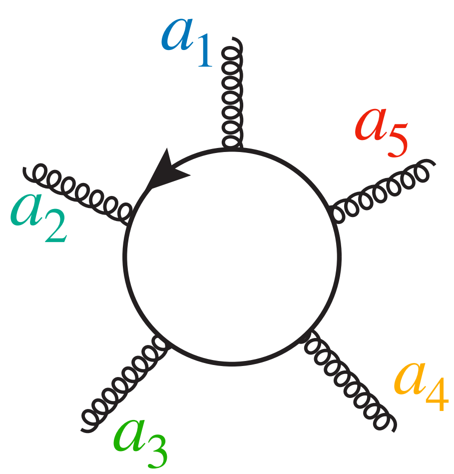


$$+ \; A_2$$

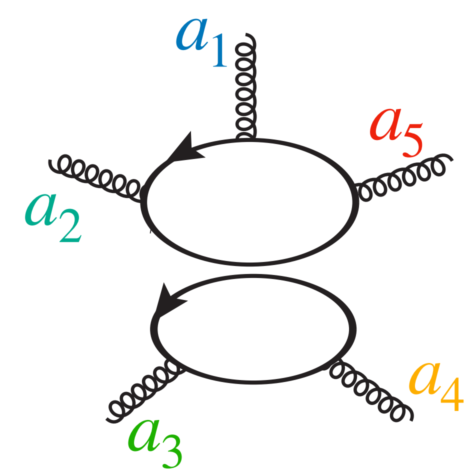


+ permutations …

$$A^{a_1 a_2 \ldots a_5} = A_1$$



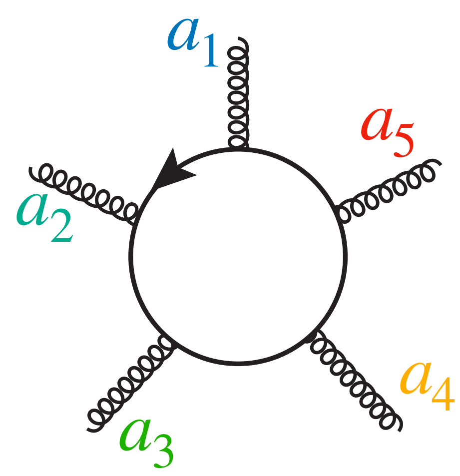
$$+ \quad A_2$$



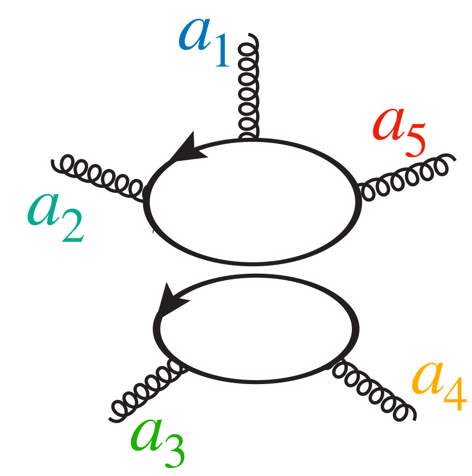
+ permutations …

$$A_1 = a_1 N_c^2 + a_2 1 + a_3 N_c n_f + a_4 N_c^{-1} n_f + a_5 n_f^2$$

$$A^{a_1 a_2 \ldots a_5} = A_1$$



$$+ \quad A_2$$

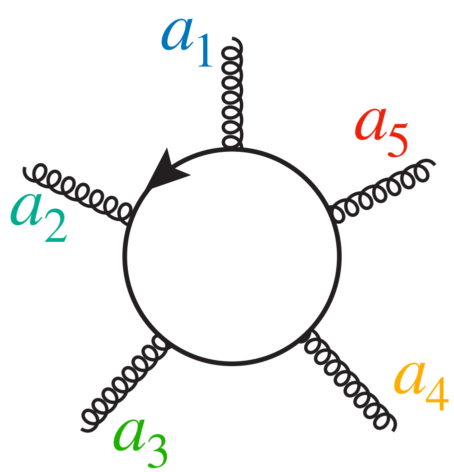


+ permutations …

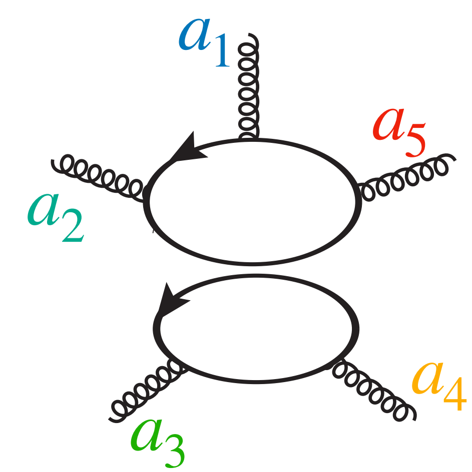
Leading colour

$$A_1 = a_1 N_c^2 + a_2 1 + a_3 N_c n_f + a_4 N_c^{-1} n_f + a_5 n_f^2$$

$$A^{a_1 a_2 \ldots a_5} = A_1$$



$$+ \quad A_2$$

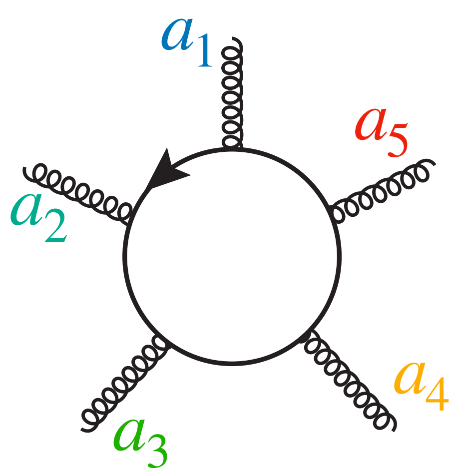


+ permutations …

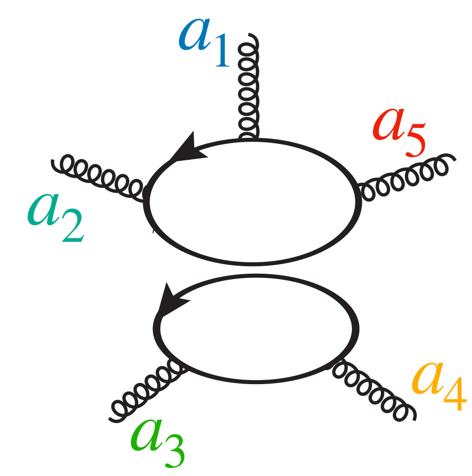
Leading colour

$$A_1 = a_1 N_c^2 + a_2 1 + a_3 N_c n_f + a_4 N_c^{-1} n_f + a_5 n_f^2$$

$$A^{a_1 a_2 \ldots a_5} = A_1$$



$$+ \quad A_2$$



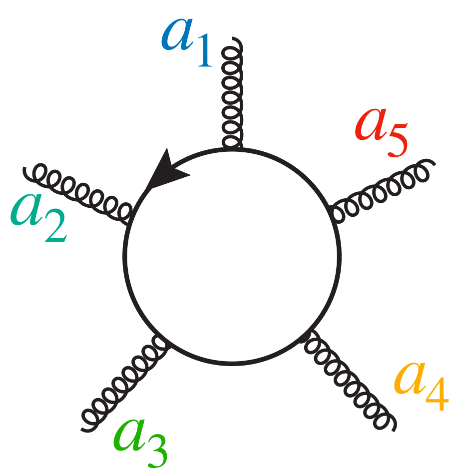
+ permutations …

Leading colour

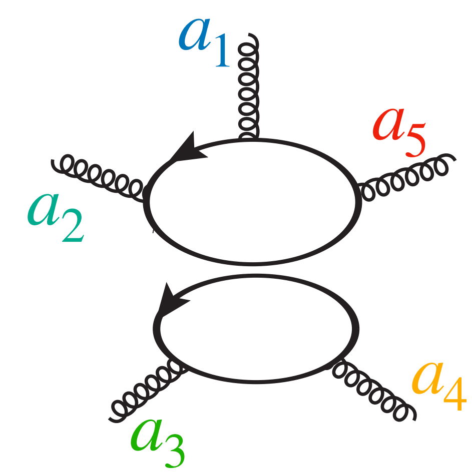
$$A_1 = a_1 N_c^2 + a_2 1 + a_3 N_c n_f + a_4 N_c^{-1} n_f + a_5 n_f^2$$

$$A_2 = b_1 N_c + b_2 n_f$$

$$A^{a_1 a_2 \ldots a_5} = A_1$$



$$+ \; A_2$$



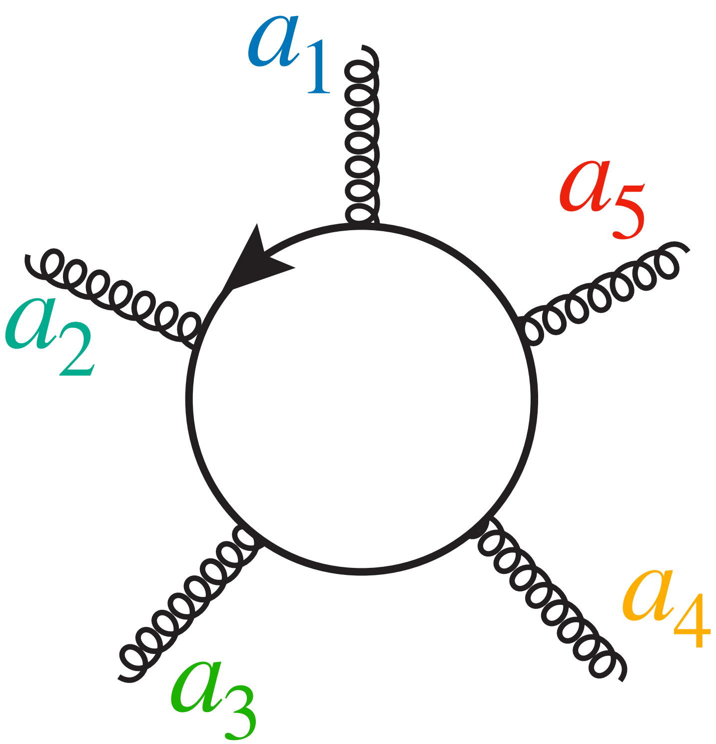
+ permutations …

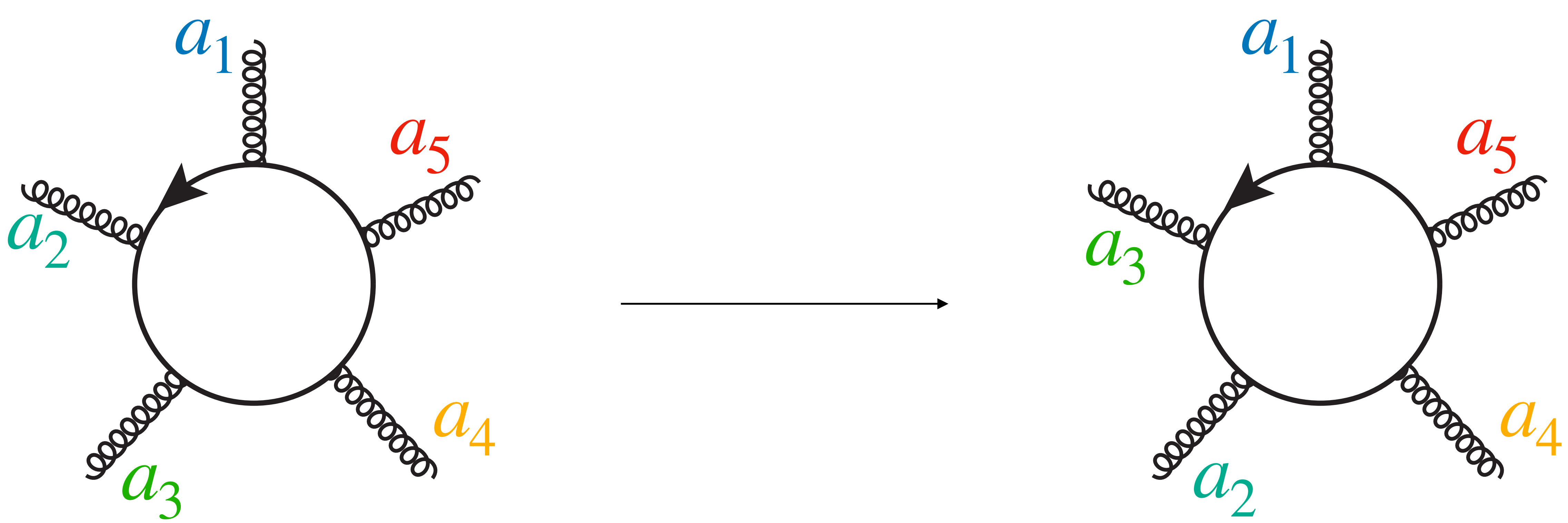
Leading colour

$$A_1 = a_1 N_c^2 + a_2 1 + a_3 N_c n_f + a_4 N_c^{-1} n_f + a_5 n_f^2$$

$$A_2 = b_1 N_c + b_2 n_f$$

95%


a_1
a_5
a_2
a_4
a_3



a_1
a_5
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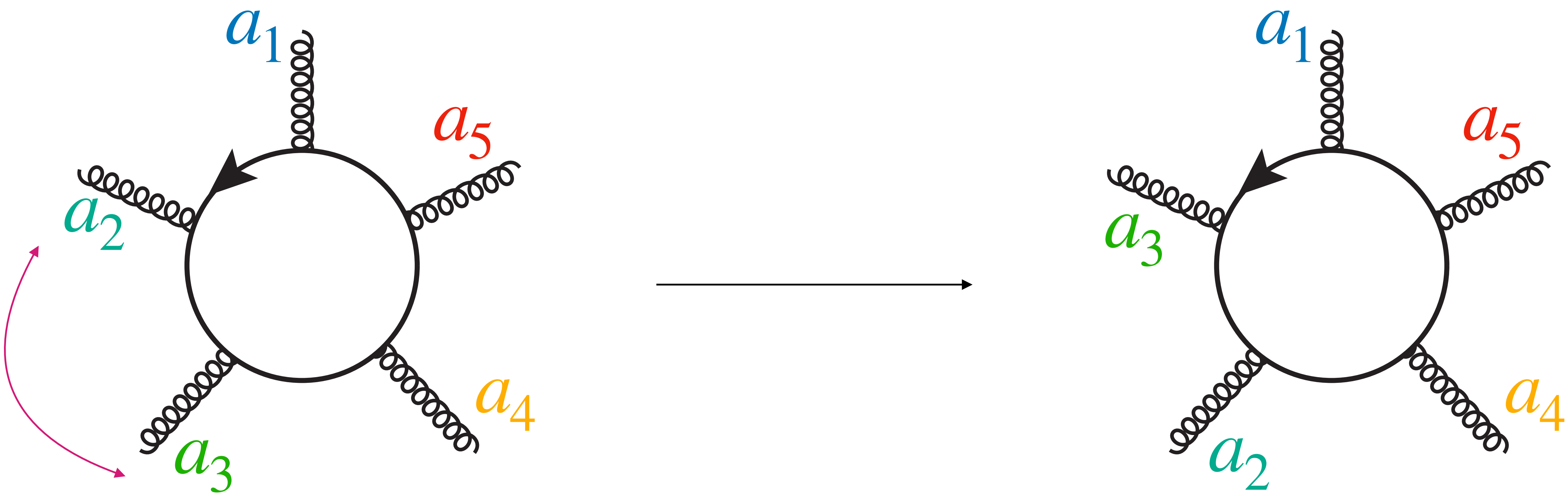
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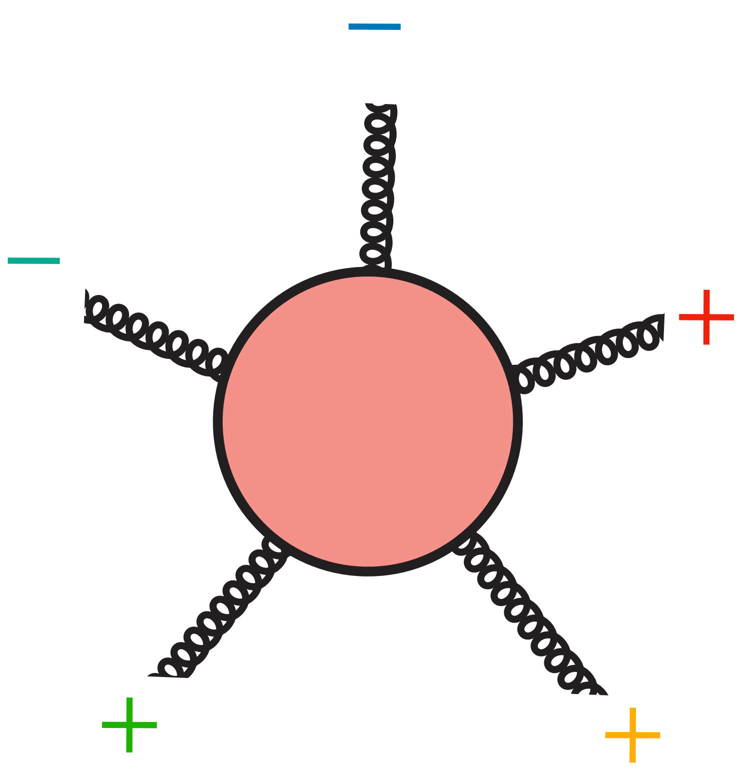
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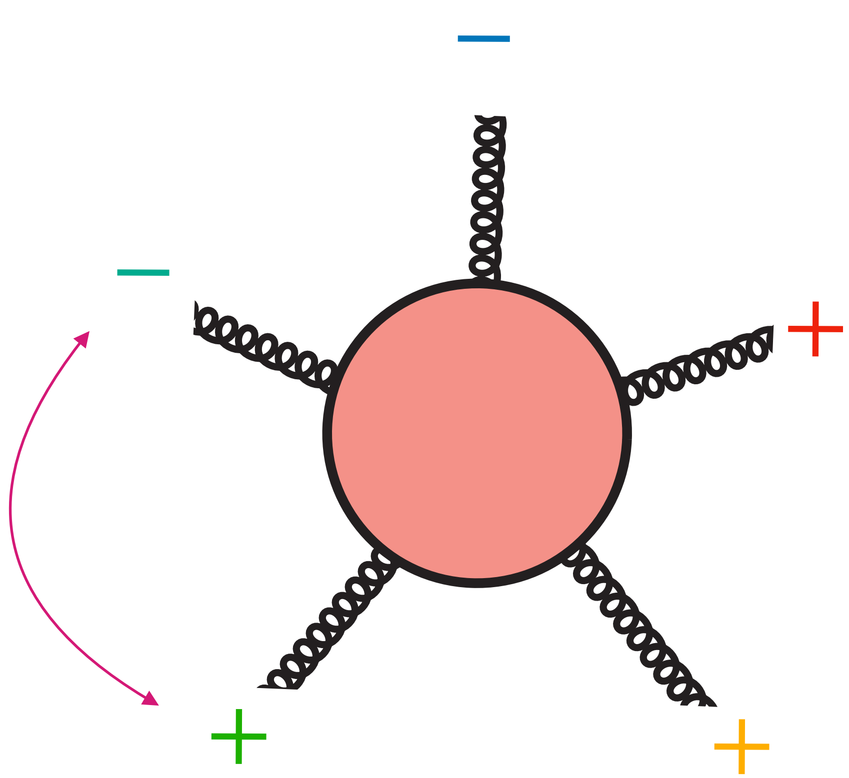

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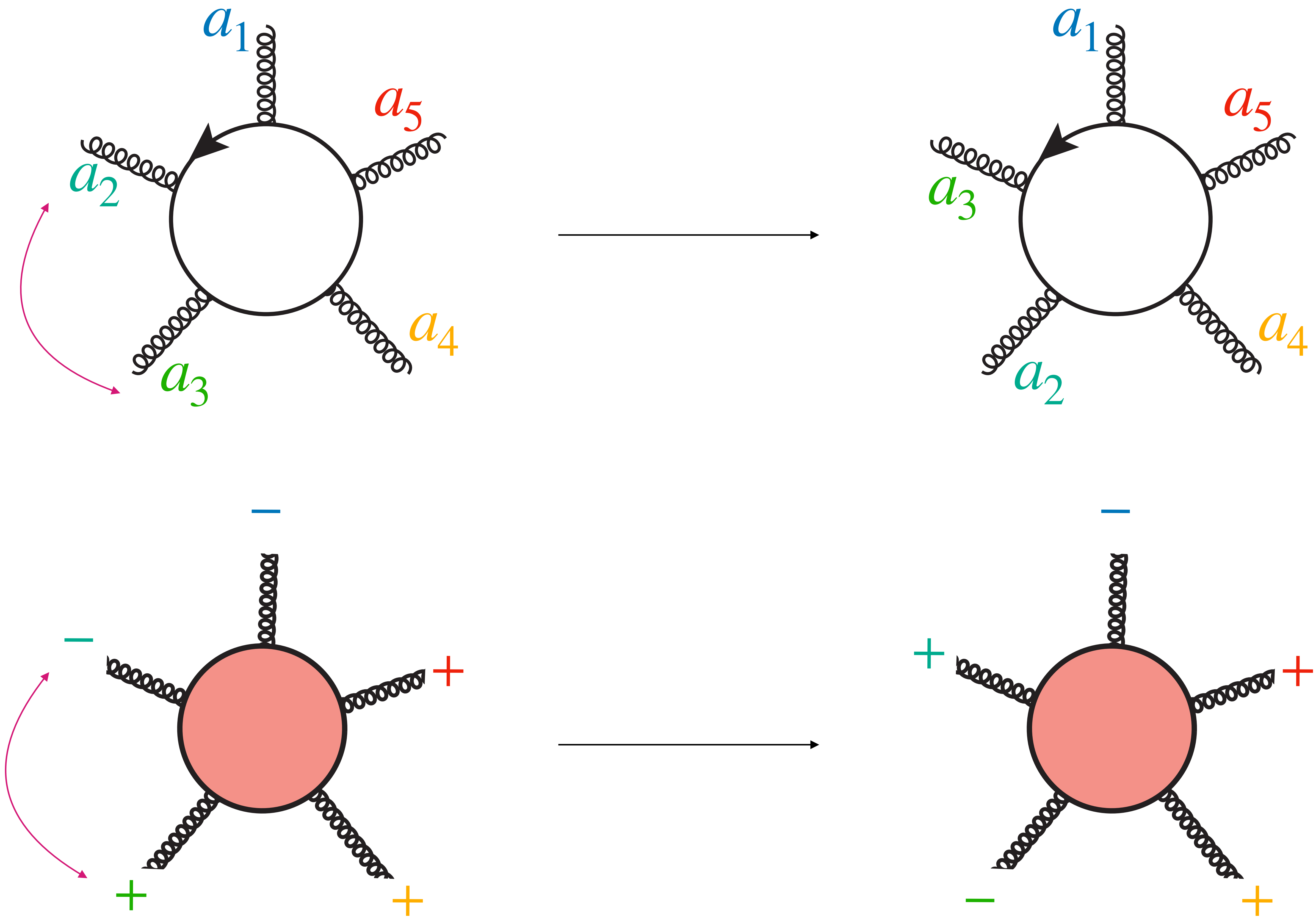

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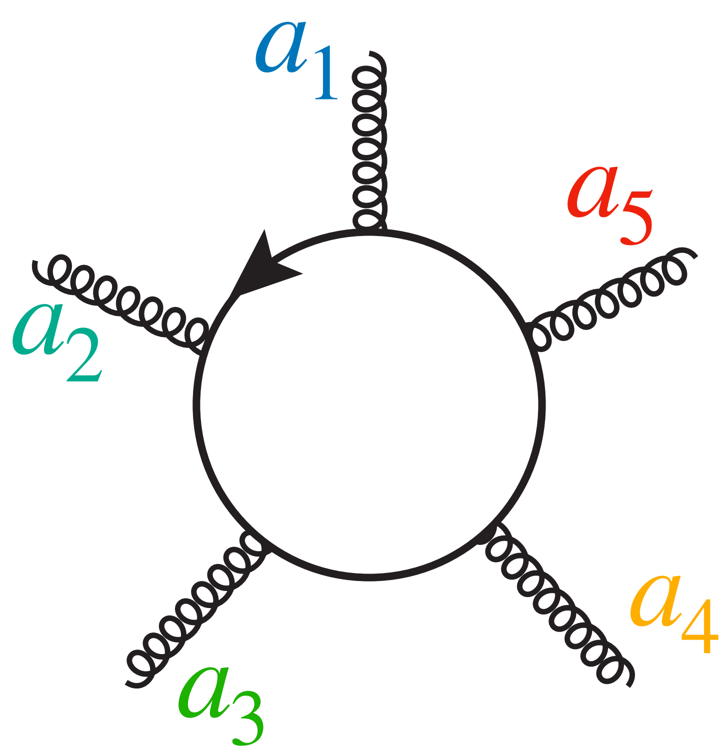
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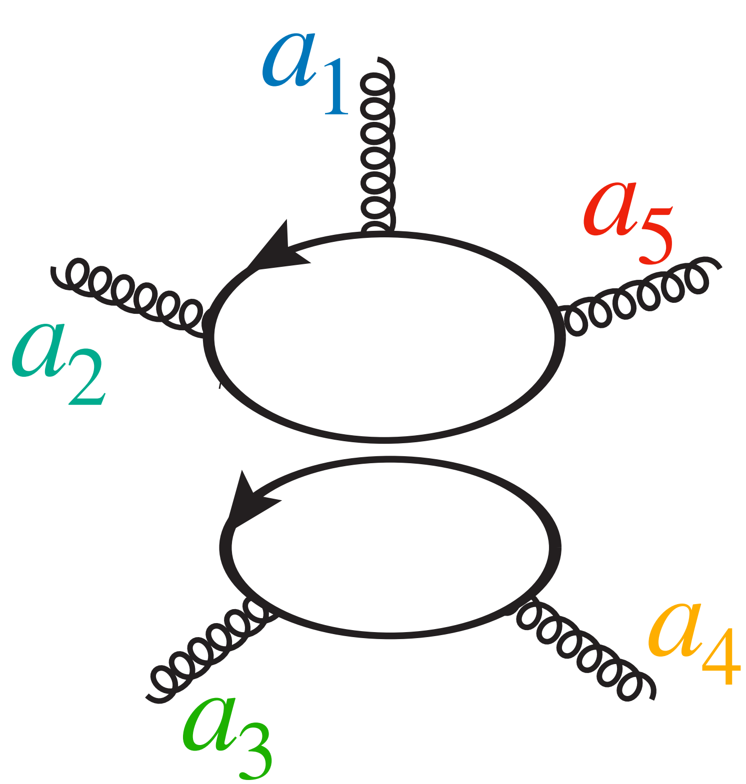
a_1
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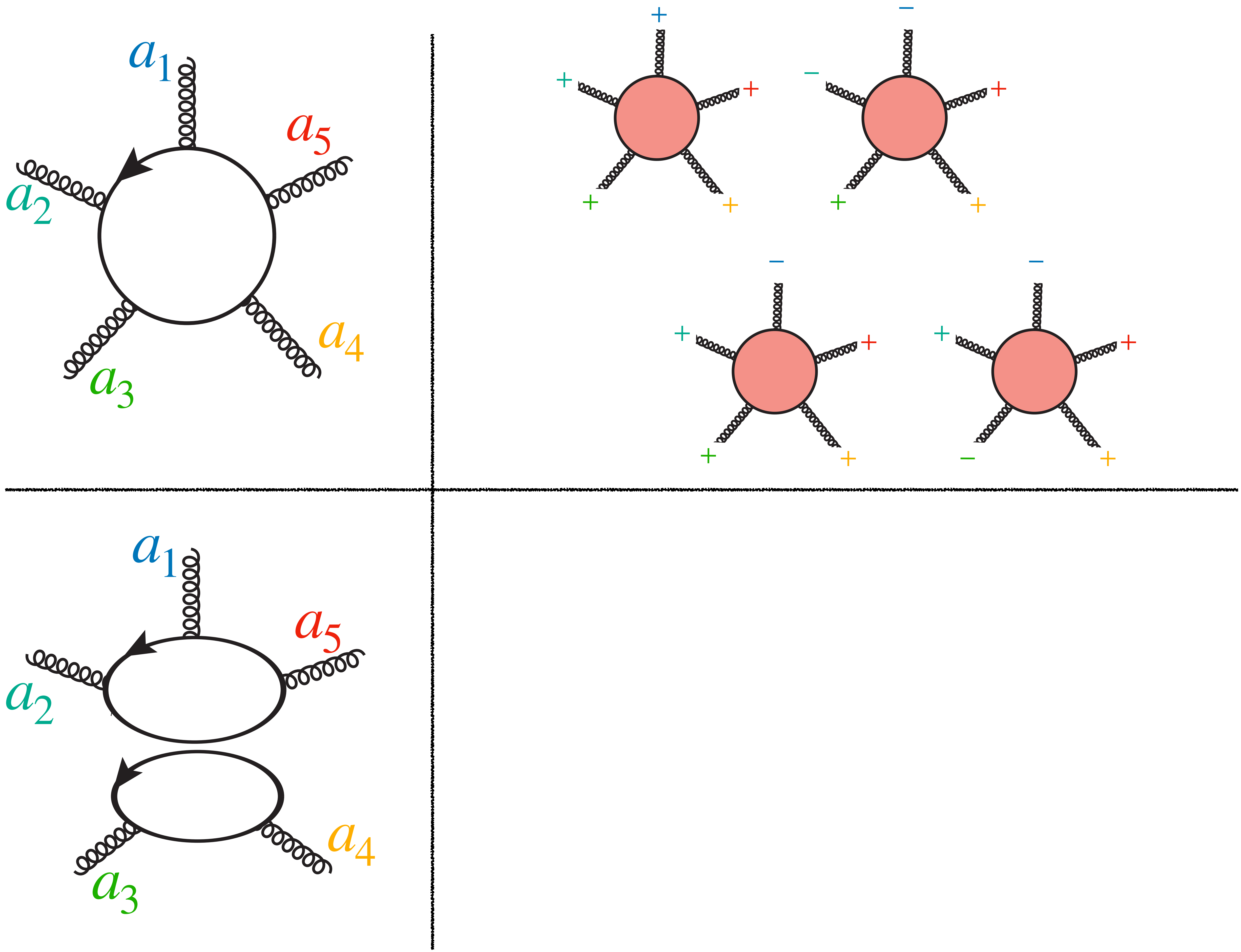

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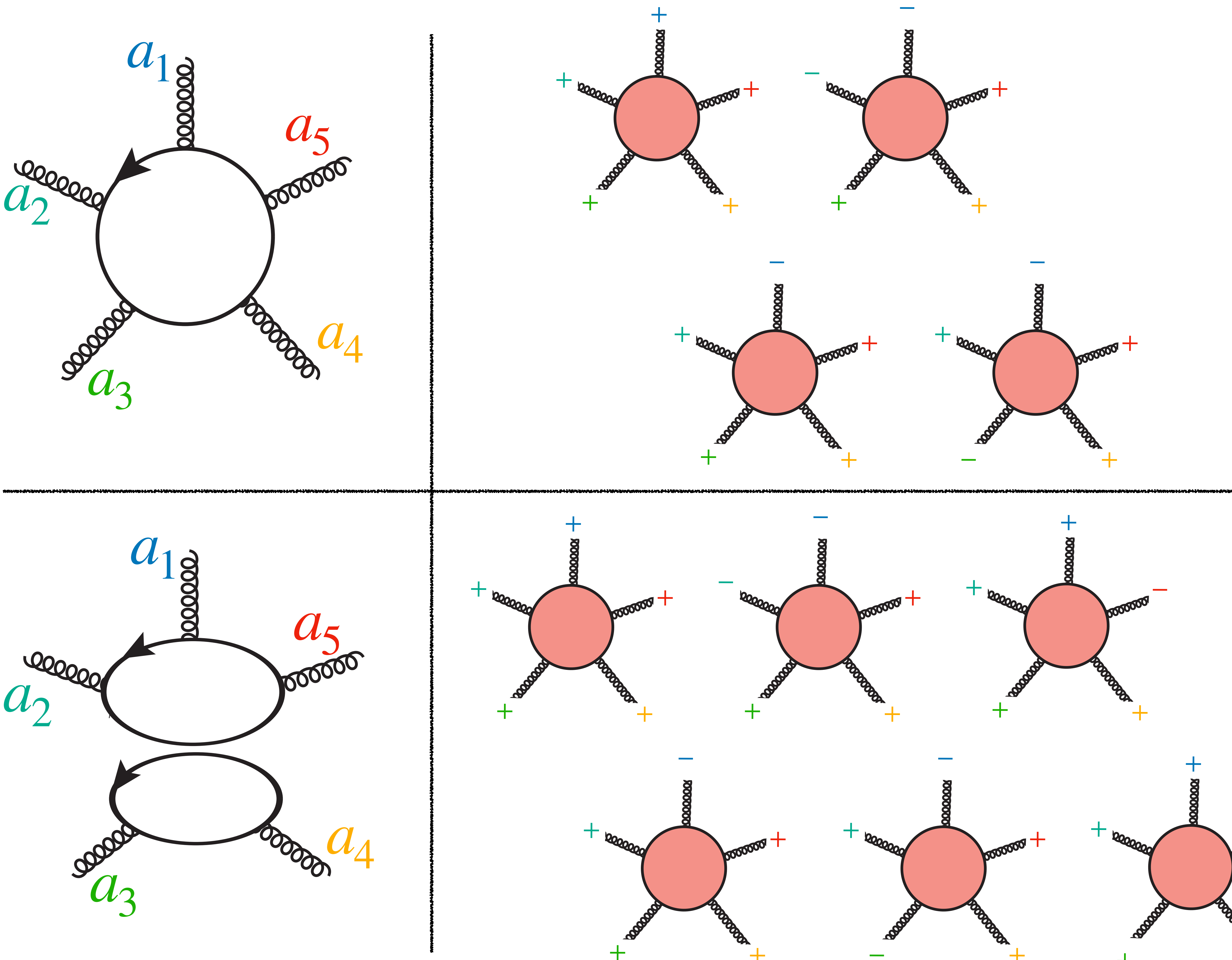

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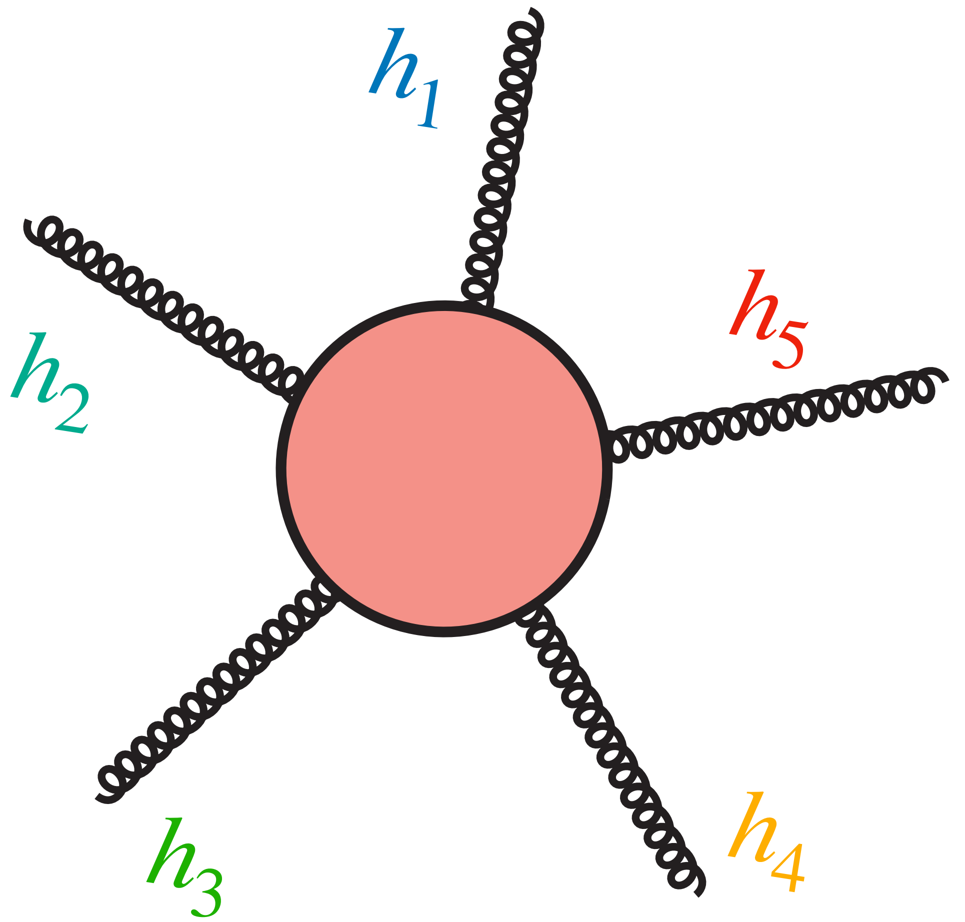
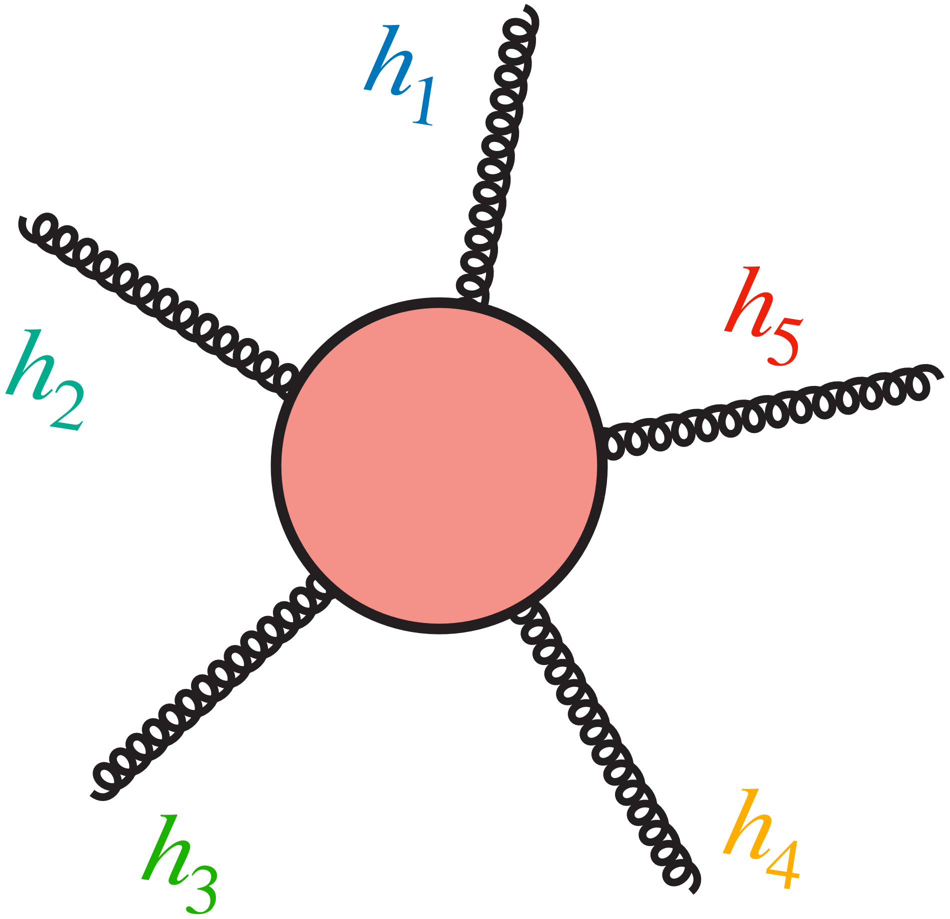
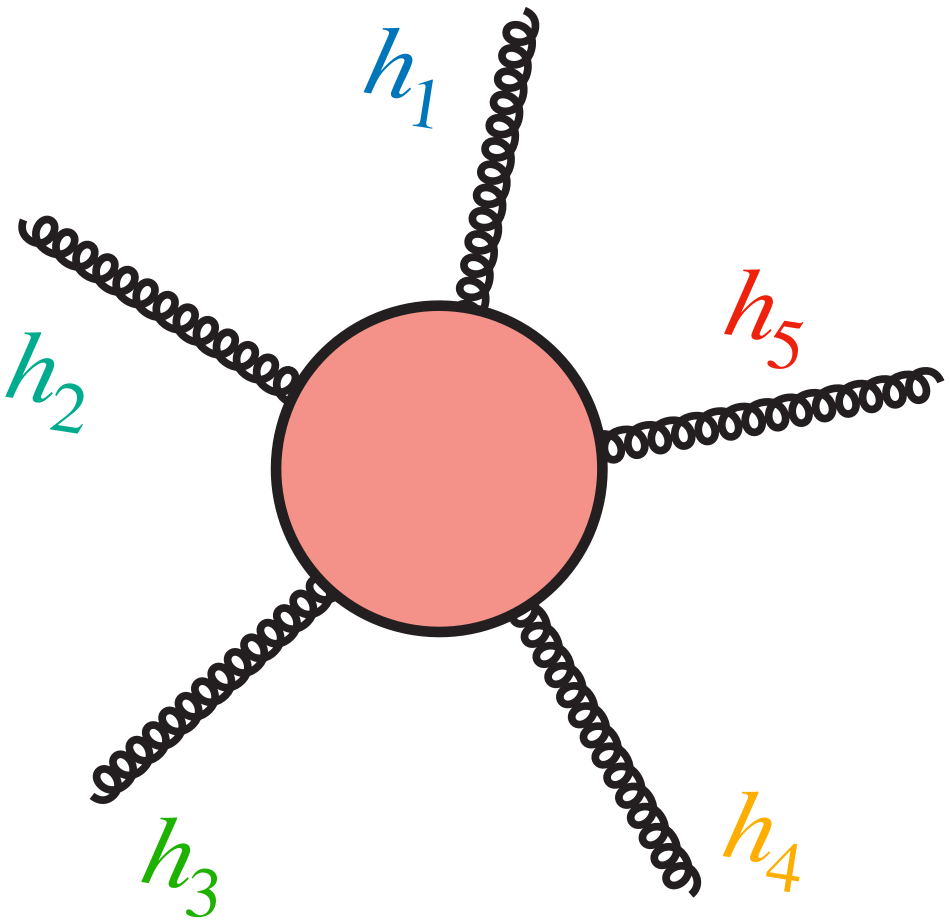
a_4
a_5



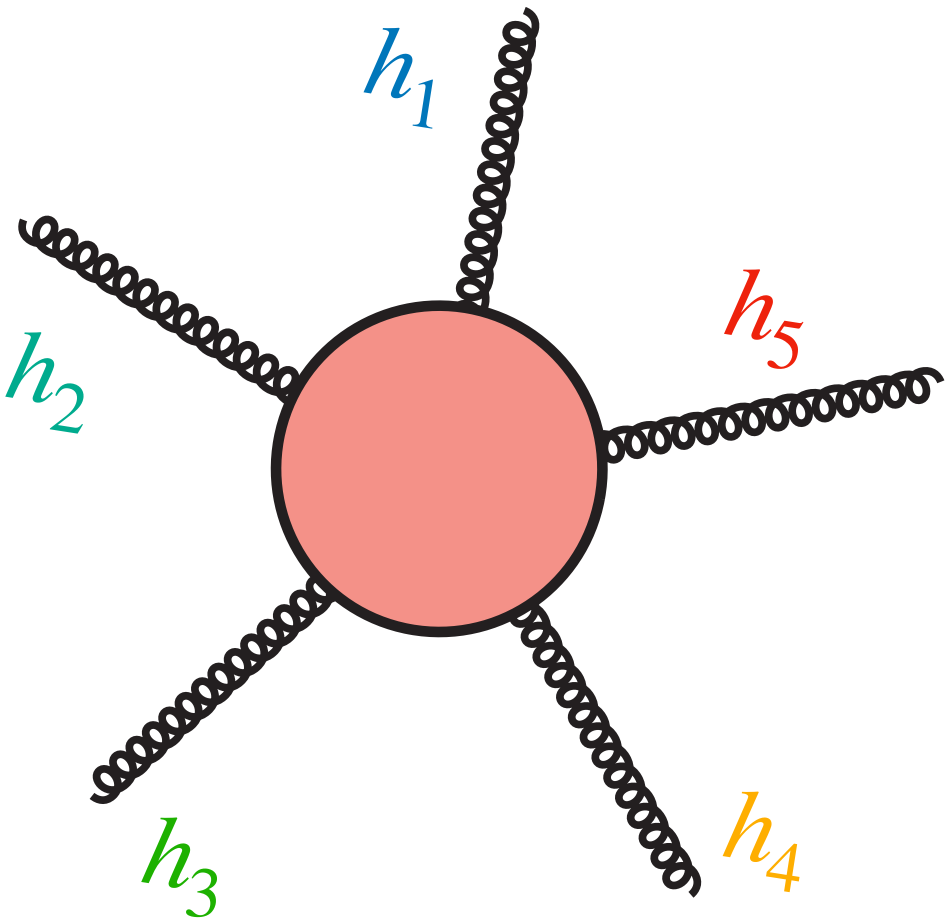
h_1
h_2
h_3
h_4
h_5



$$\mathcal{A}^{h_1,h_2,\ldots,h_5} =$$

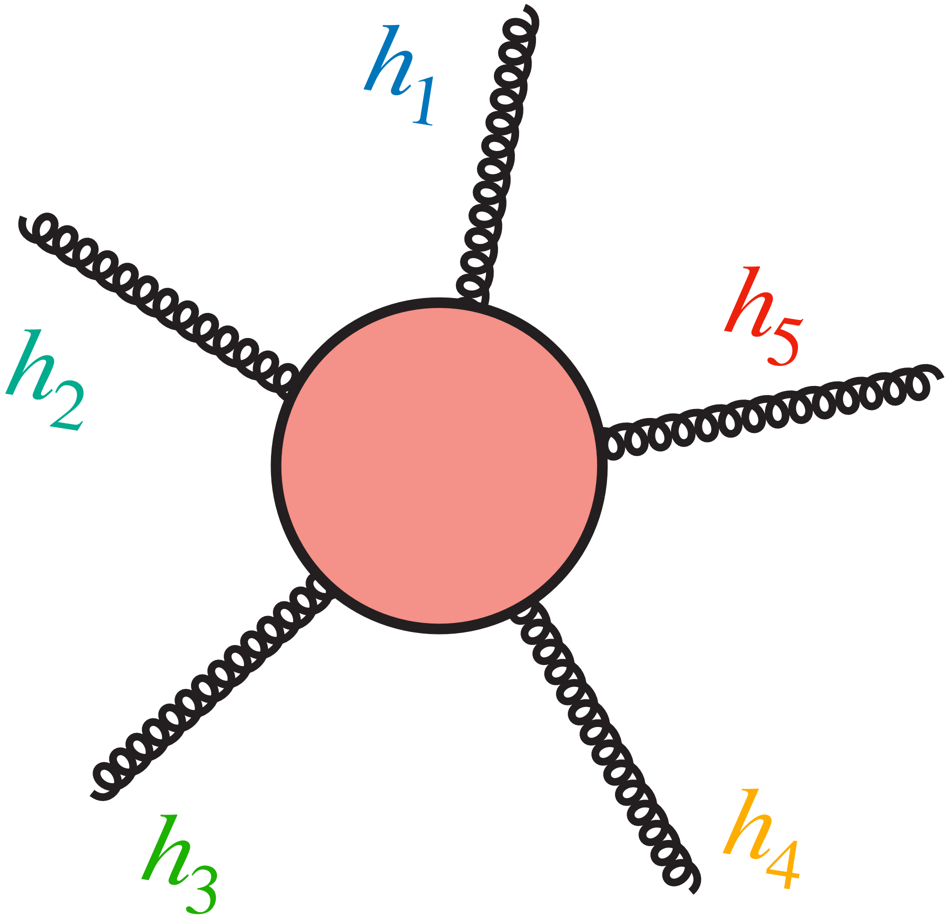


$$\mathcal{A}^{h_1,h_2,\ldots,h_5} = \mathcal{A}^{\mu_1,\mu_2,\ldots,\mu_5}\,\epsilon(p_1)_{\mu_1}\ldots\epsilon(p_5)_{\mu_5}$$



$$\mathcal{A}^{h_1,h_2,\ldots,h_5} = \underbrace{\mathcal{A}^{\mu_1,\mu_2,\ldots,\mu_5}}_{D=4-2\epsilon} \underbrace{\epsilon(p_1)_{\mu_1} \ldots \epsilon(p_5)_{\mu_5}}_{D=4}$$

't Hooft-Veltman
scheme



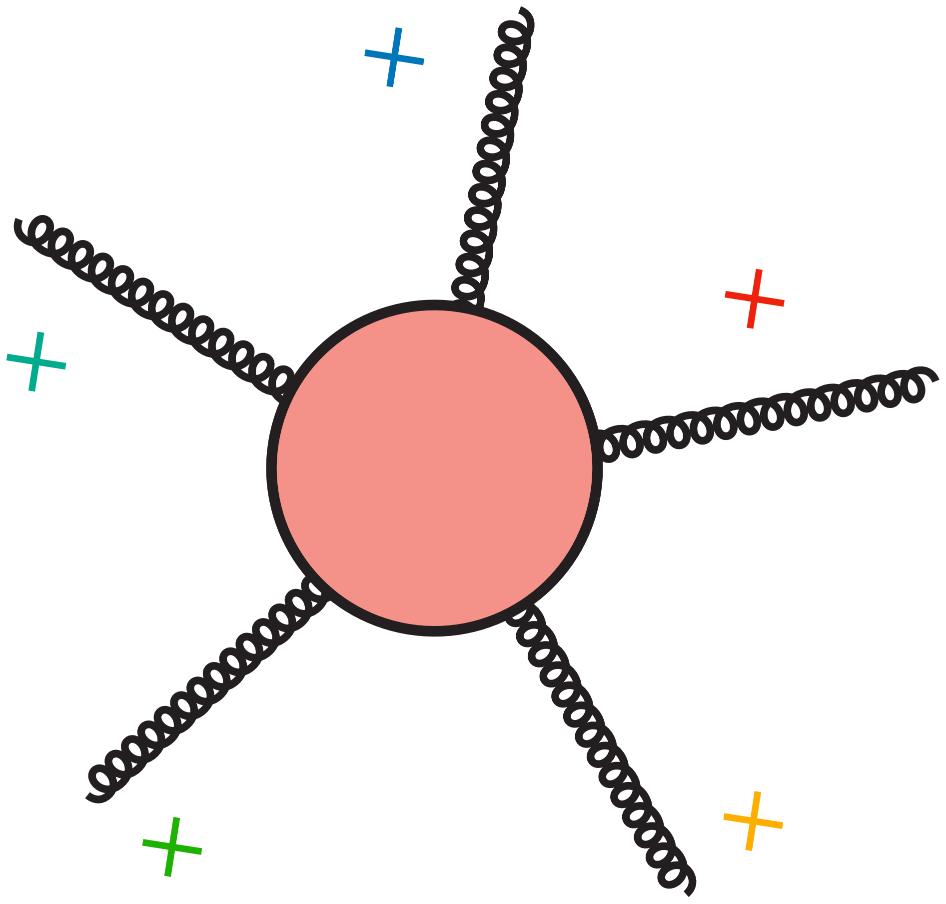
$$\mathcal{A}^{h_1,h_2,\ldots,h_5} = \underbrace{\mathcal{A}^{\mu_1,\mu_2,\ldots,\mu_5}}_{D=4-2\epsilon} \underbrace{\epsilon(p_1)_{\mu_1}\ldots\epsilon(p_5)_{\mu_5}}_{D=4}$$

't Hooft-Veltman scheme

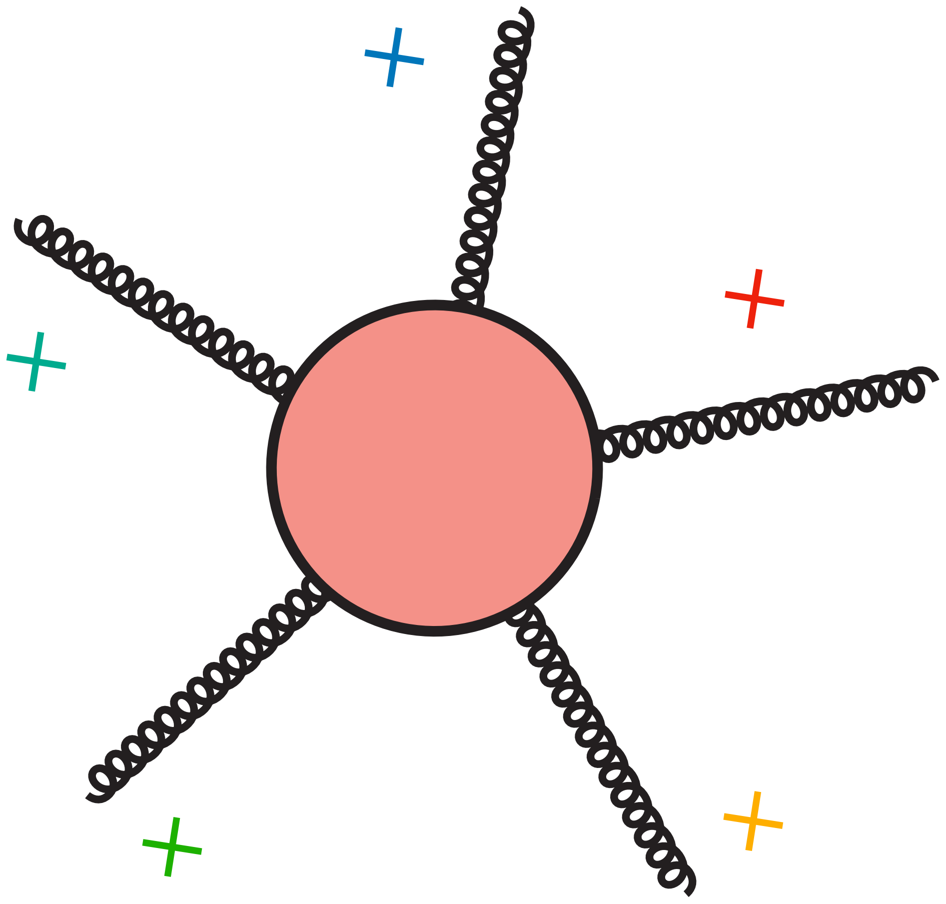
Direct Computation

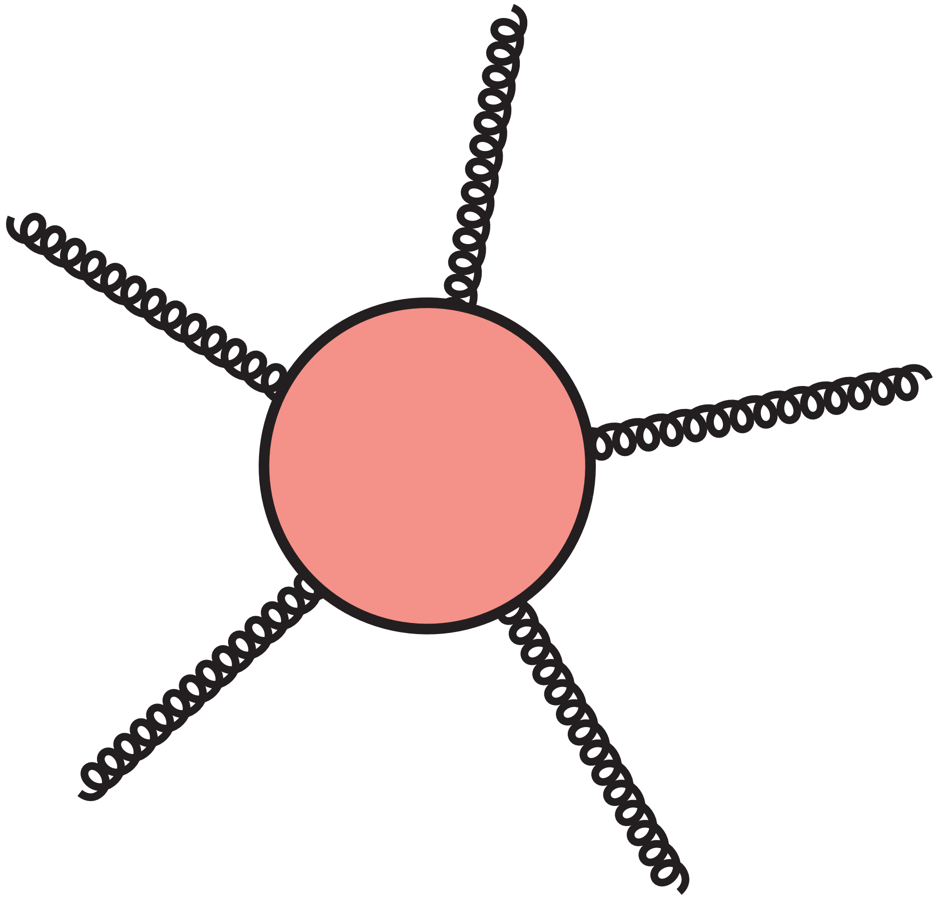
$$\epsilon_+^\mu = \frac{1}{\sqrt{2}}\frac{[q\mu i\rangle}{[iq]} \qquad \epsilon_-^\mu = \frac{1}{\sqrt{2}}\frac{\langle q\mu i]}{\langle iq\rangle}$$

Example

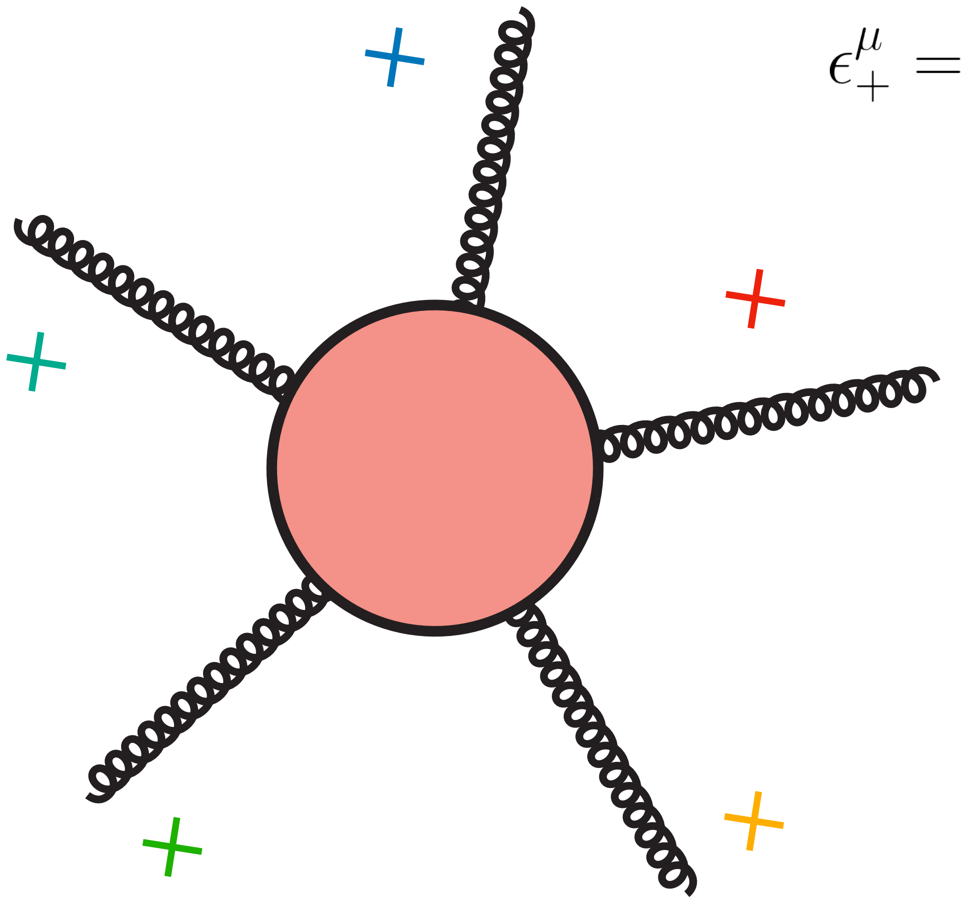


Example

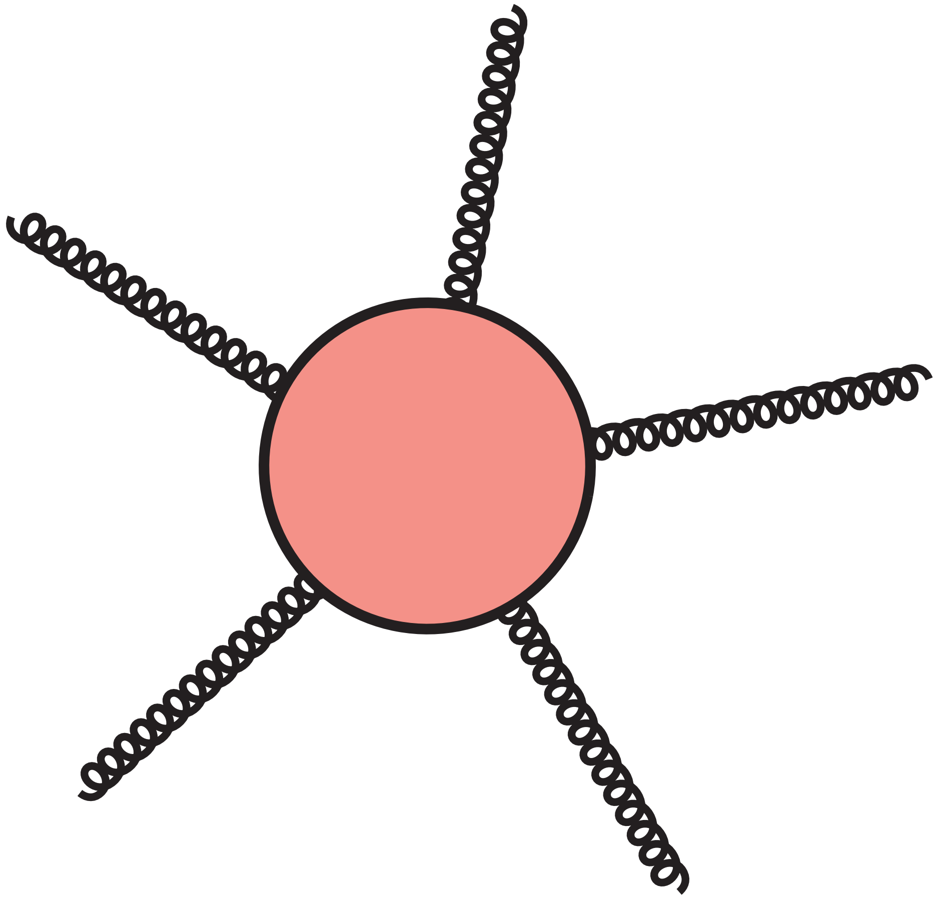




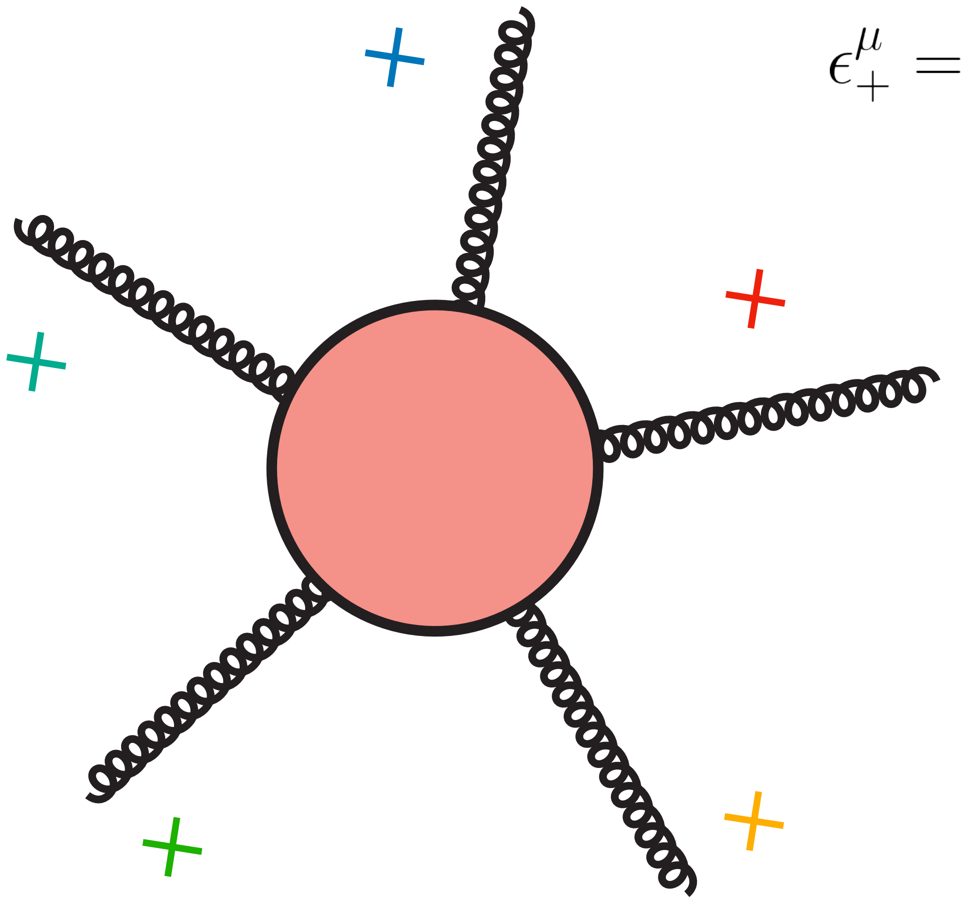
Example



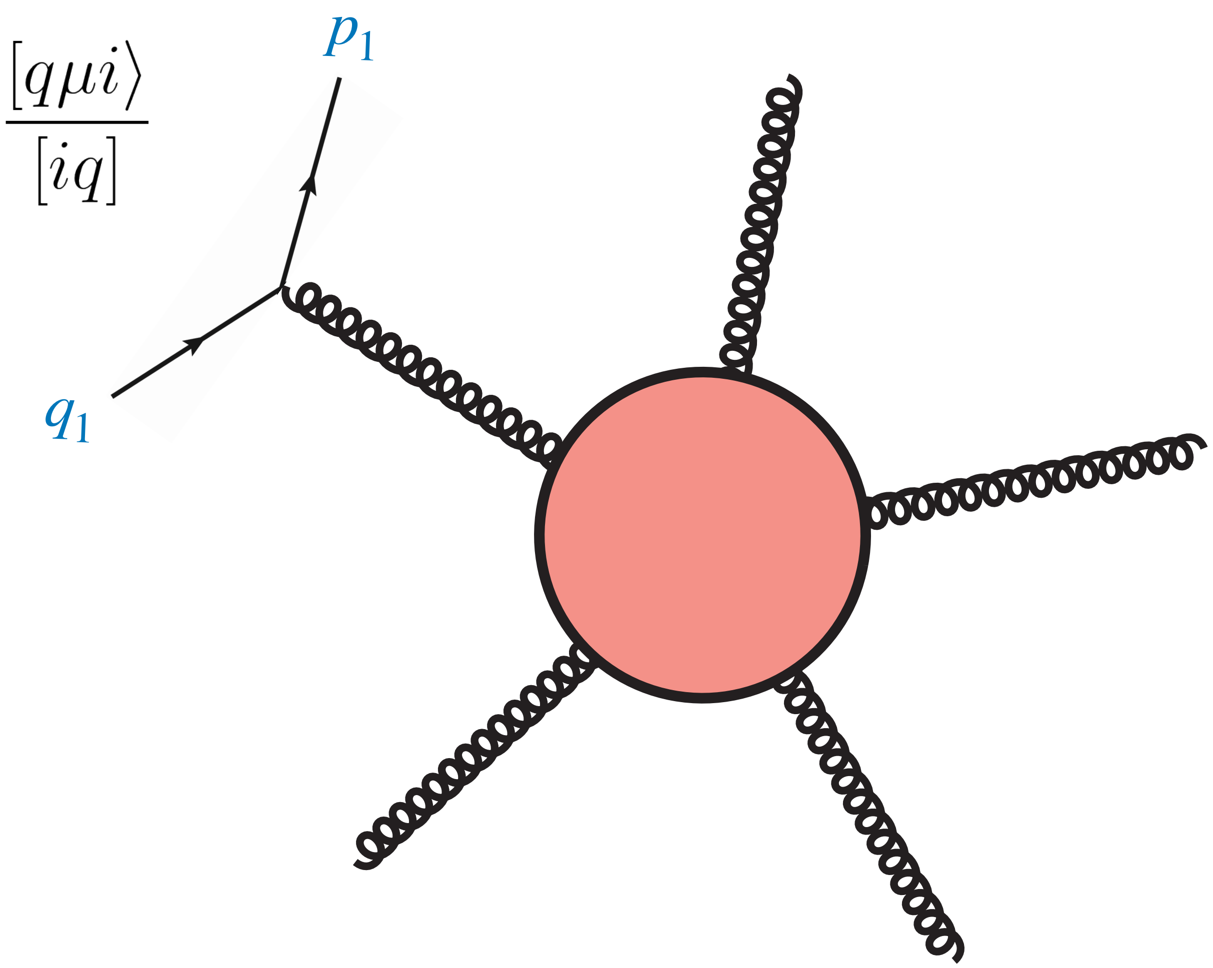
$$\epsilon_+^\mu = \frac{1}{\sqrt{2}}\frac{[q\mu i\rangle}{[iq]}$$



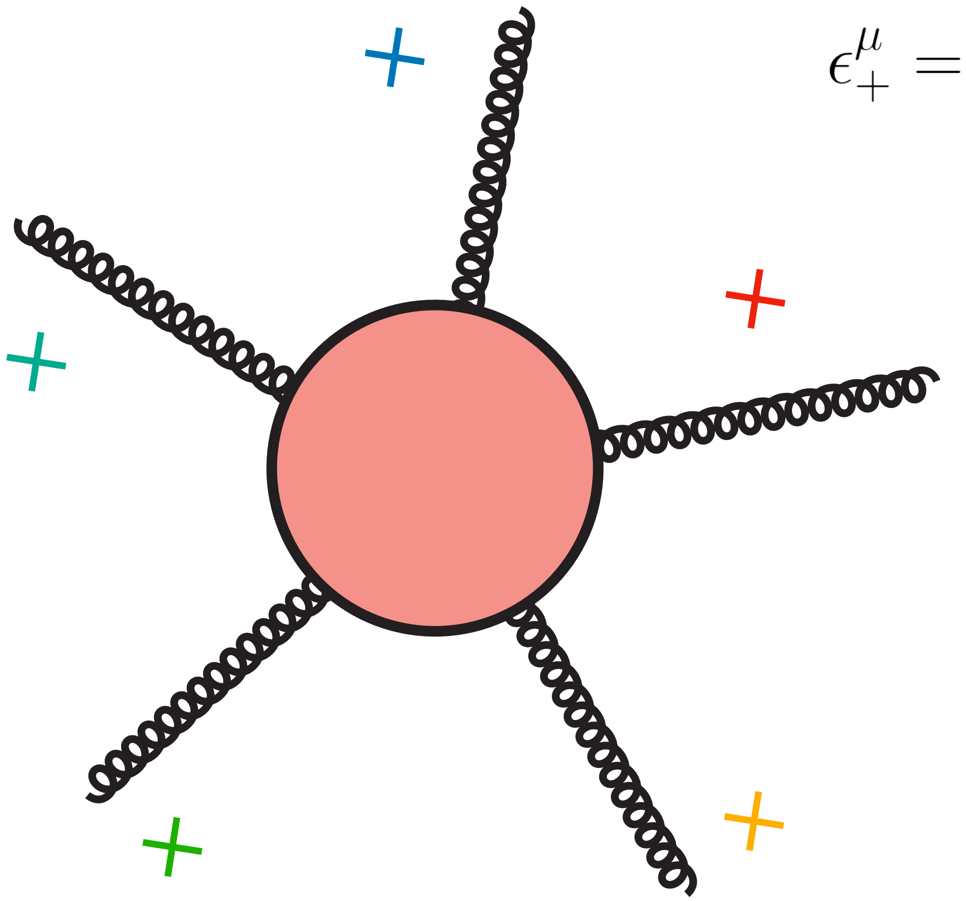
Example



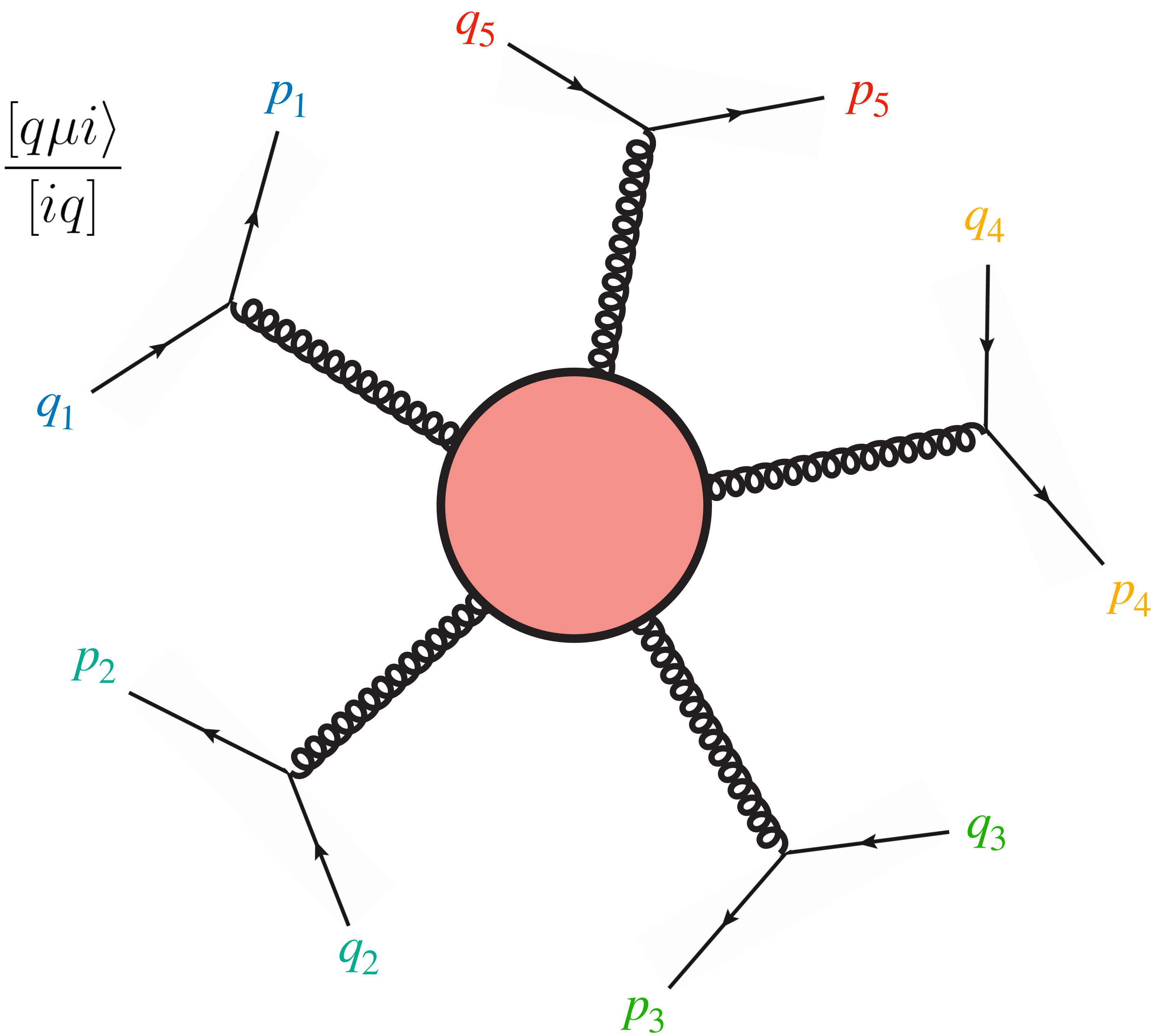
$$\epsilon_+^\mu = \frac{1}{\sqrt{2}} \frac{[q\mu i\rangle}{[iq]}$$



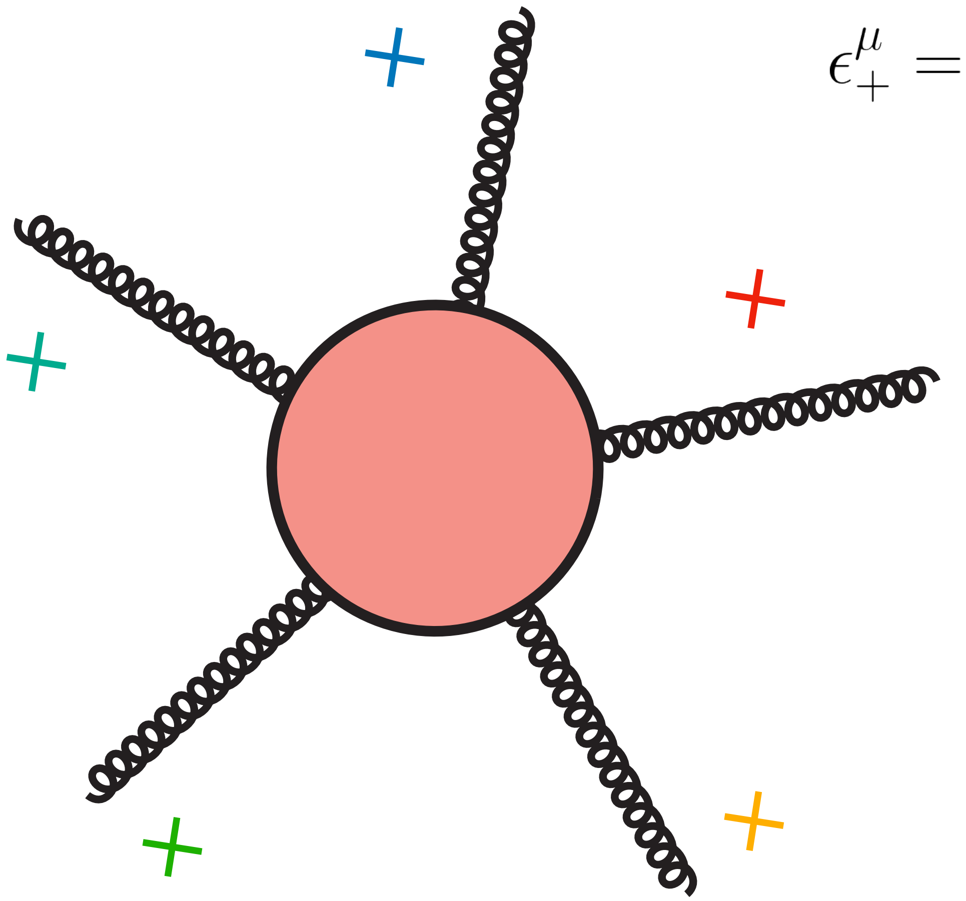
Example



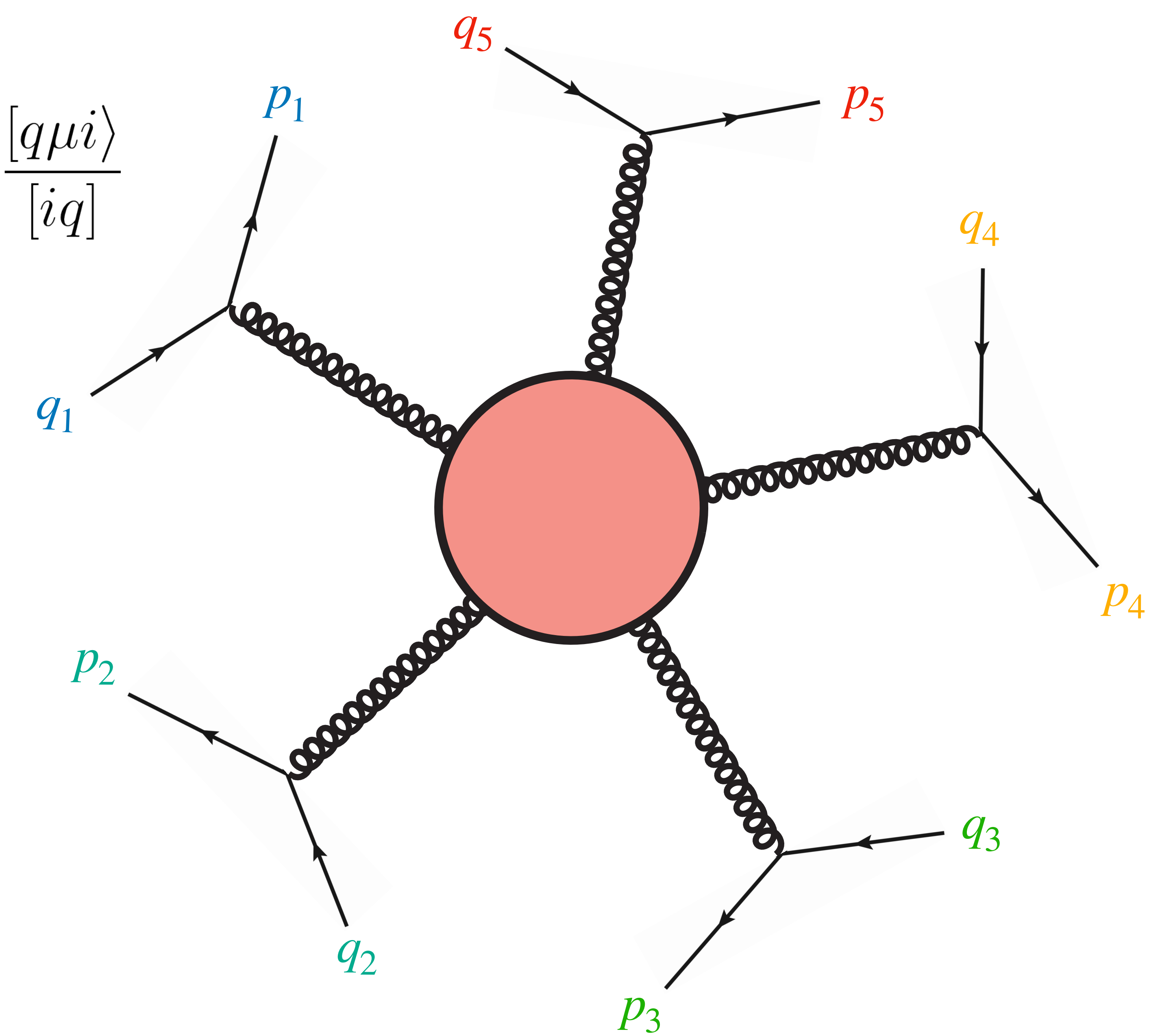
$$\epsilon_+^\mu = \frac{1}{\sqrt{2}} \frac{[q\mu i\rangle}{[iq]}$$



Example

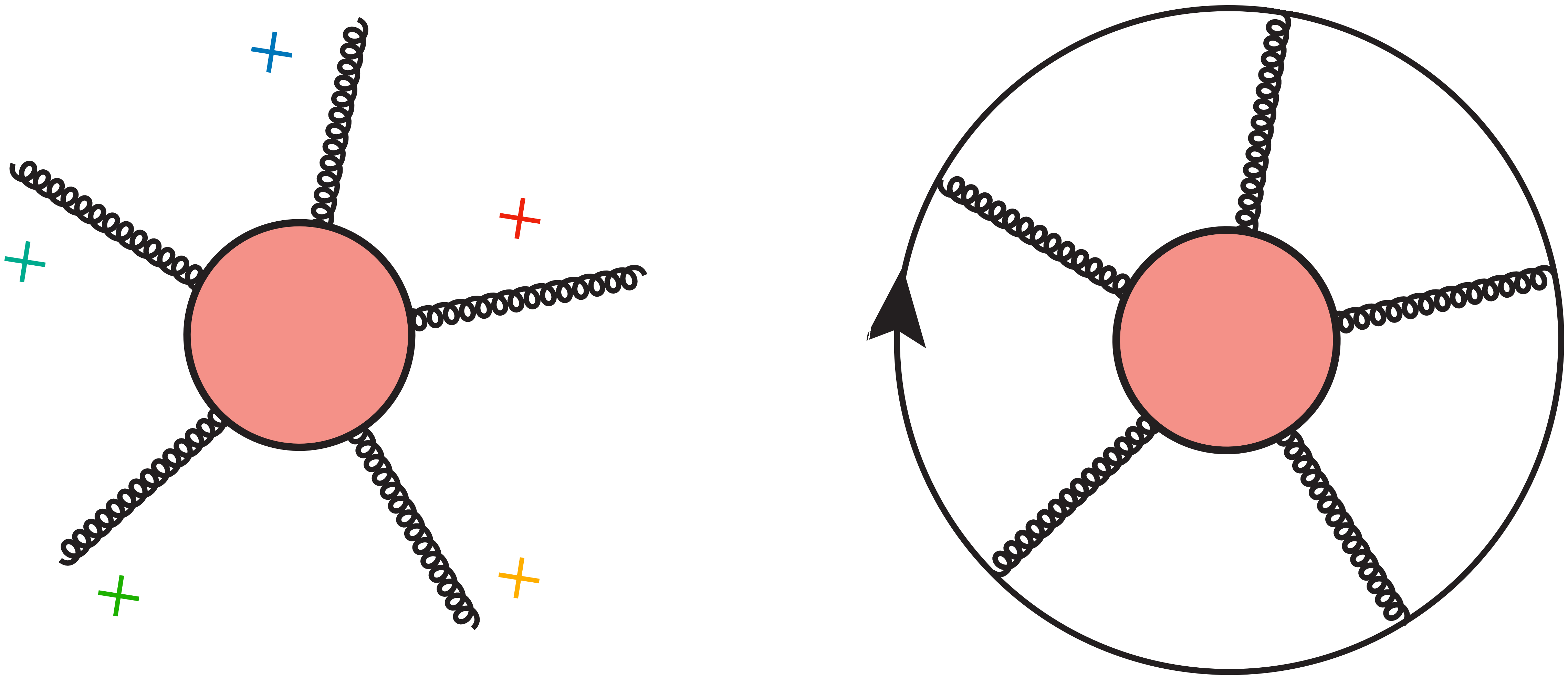


$$\epsilon_+^\mu = \frac{1}{\sqrt{2}} \frac{[q\mu i\rangle}{[iq]}$$



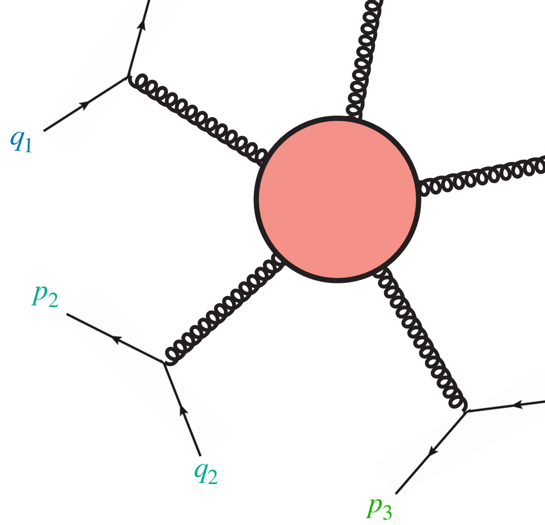
$$\epsilon_+^{\mu_1}\epsilon_+^{\mu_2}\epsilon_+^{\mu_3}\epsilon_+^{\mu_4}\epsilon_+^{\mu_5} = \frac{1}{2^{5/2}} \frac{[q_1\mu_1 1\rangle}{[q_1 1]} \frac{[q_2\mu_2 2\rangle}{[q_2 2]} \frac{[q_3\mu_3 3\rangle}{[q_3 3]} \frac{[q_4\mu_4 4\rangle}{[q_4 4]} \frac{[q_5\mu_5 5\rangle}{[q_5 5]}$$

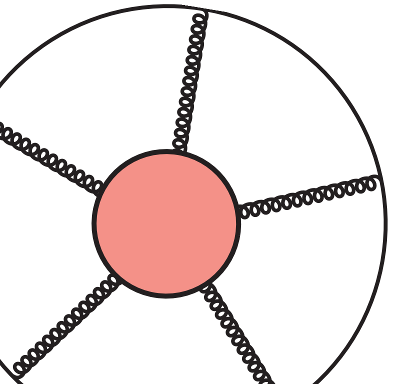
Example



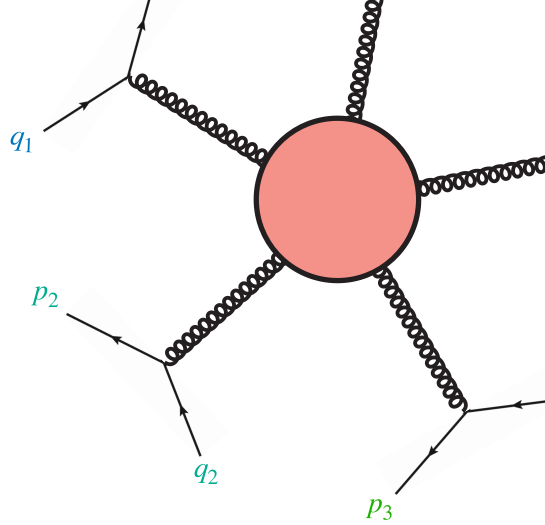
$$\epsilon_+^{\mu_1}\epsilon_+^{\mu_2}\epsilon_+^{\mu_3}\epsilon_+^{\mu_4}\epsilon_+^{\mu_5} = \frac{1}{2^{7/2}}\frac{\mathrm{Tr}\left[(1-\gamma_5)\gamma^{\mu_1}\not{p}_1\gamma^{\mu_2}\not{p}_2\gamma^{\mu_3}\not{p}_3\gamma^{\mu_4}\not{p}_4\gamma^{\mu_5}\not{p}_5\right]}{[12][23][34][45][51]}$$

Why?

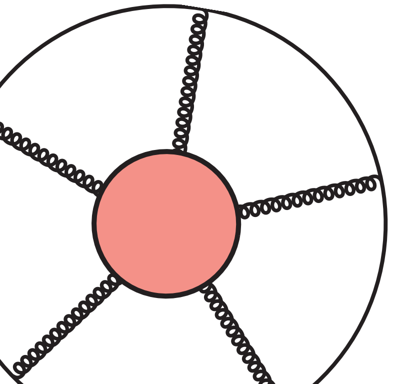




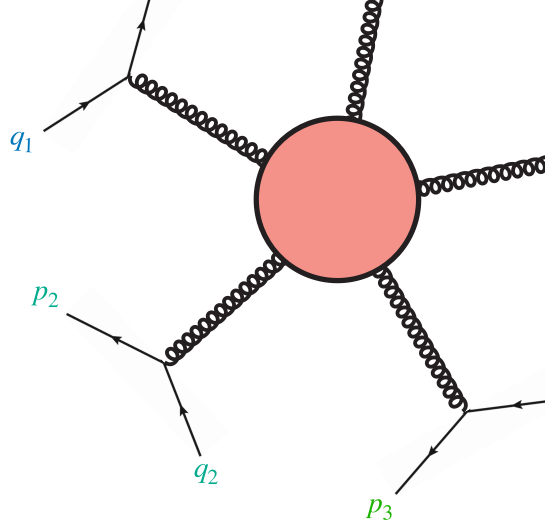
Why?



avoid computation of projectors

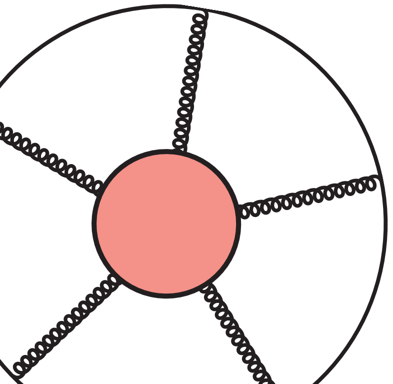


Why?

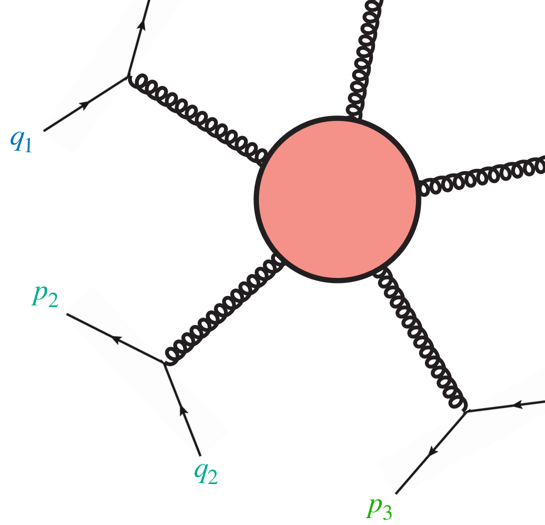


avoid computation of projectors

better handle on complexity



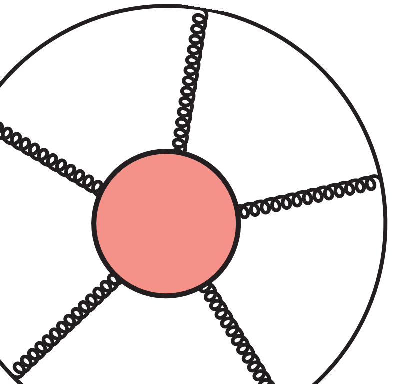
Why?



avoid computation of projectors

better handle on complexity

reference choice ↔ symmetry of amplitude



$$\mathcal{H} = \sum_{i,c} R_{i,c}\, \mathcal{I}_i\, \mathcal{C}_c$$

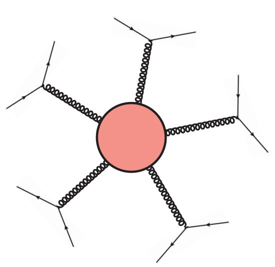
Feynman diagrams

$$\mathcal{H} = \sum_{i,c} R_{i,c}\, \mathcal{I}_i\, \mathcal{C}_c$$

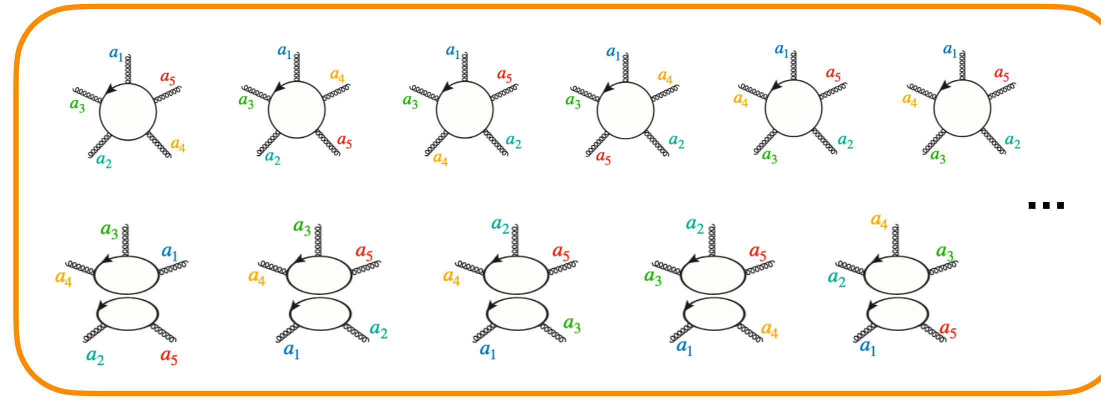
Feynman diagrams → helicity projection

$$\mathcal{H} = \sum_{i,c} R_{i,c}\, \mathcal{I}_i\, C_c$$

direct evaluation

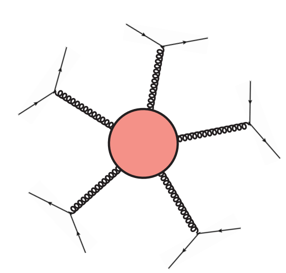


Feynman diagrams → helicity projection → colour decomposition

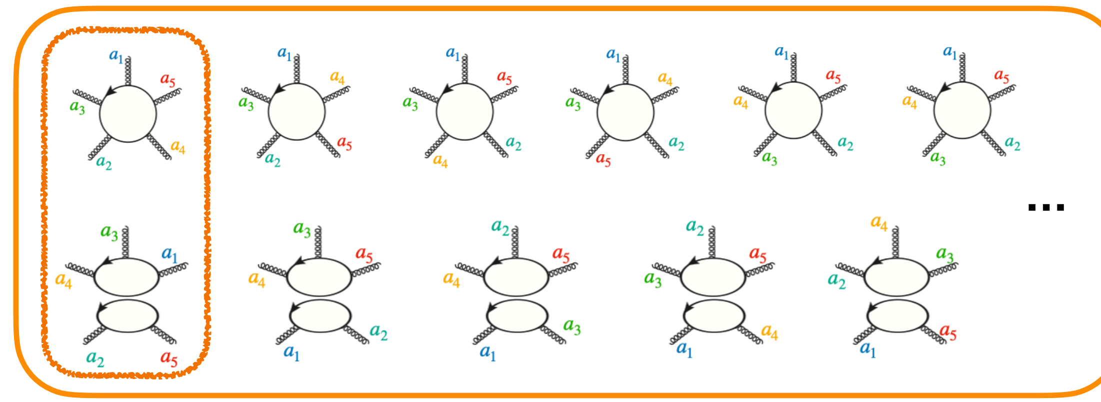


$$\mathcal{H} = \sum_{i,c} R_{i,c}\, \mathcal{I}_i\, C_c$$

direct evaluation

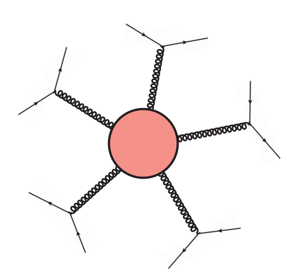


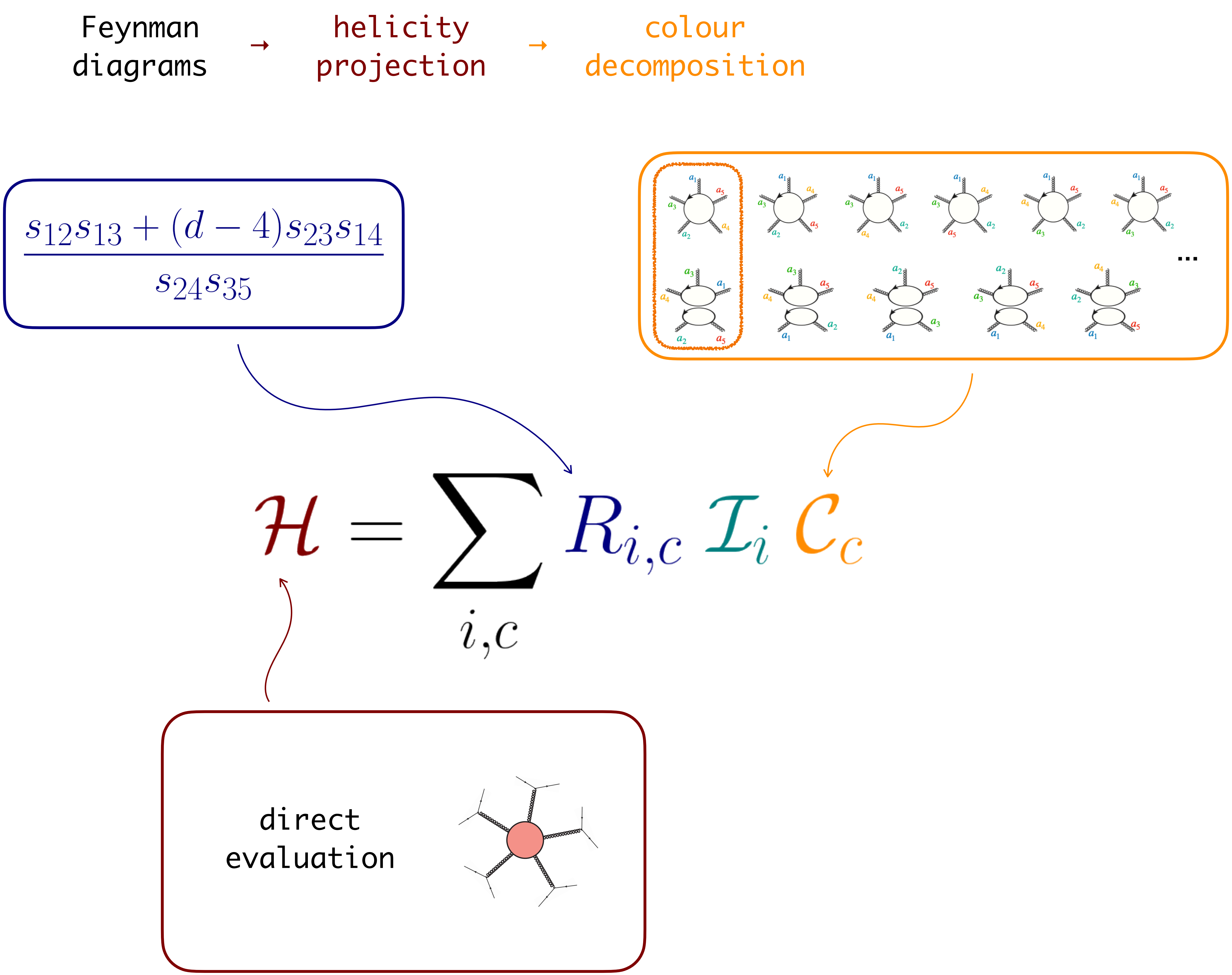
Feynman diagrams → helicity projection → colour decomposition

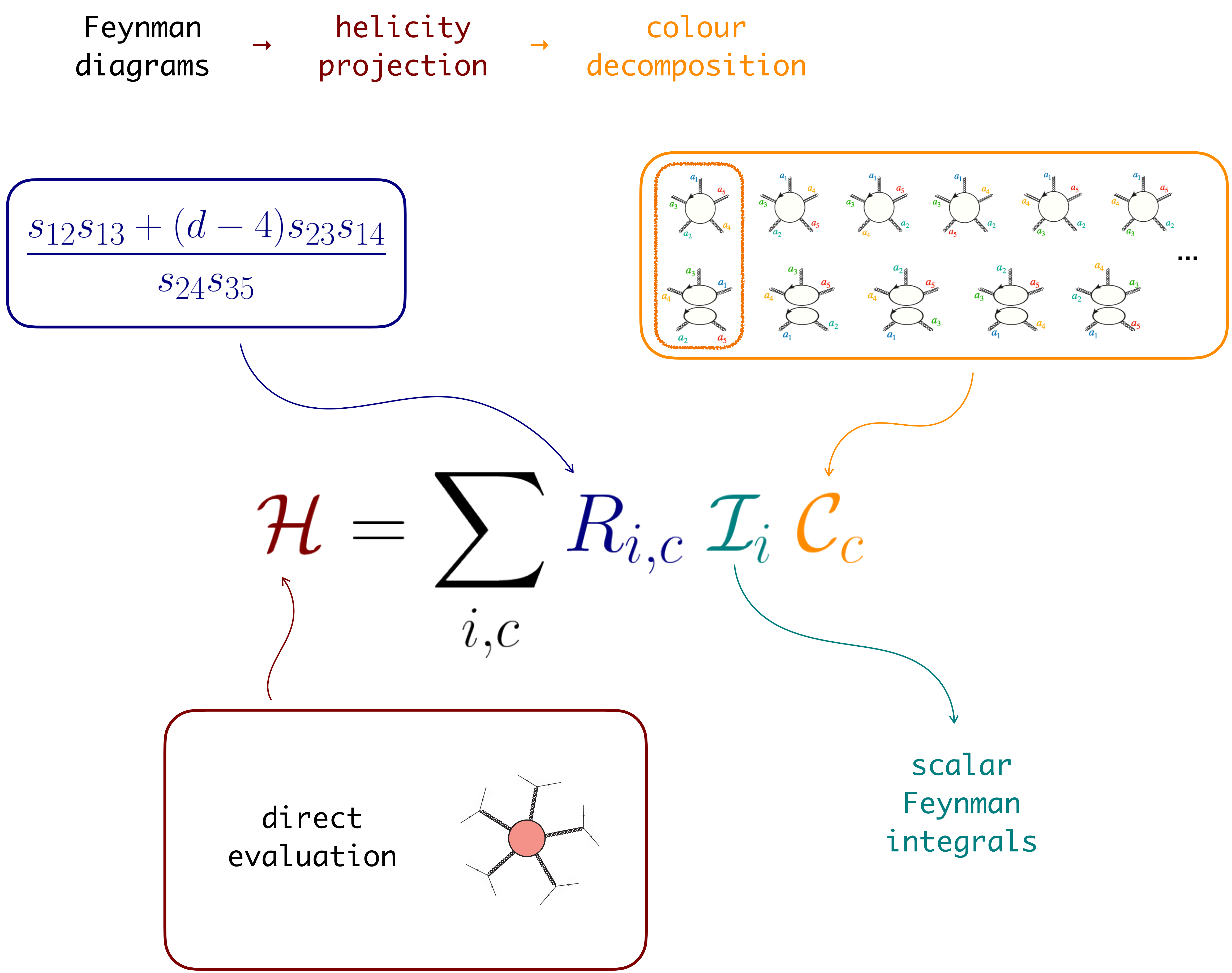


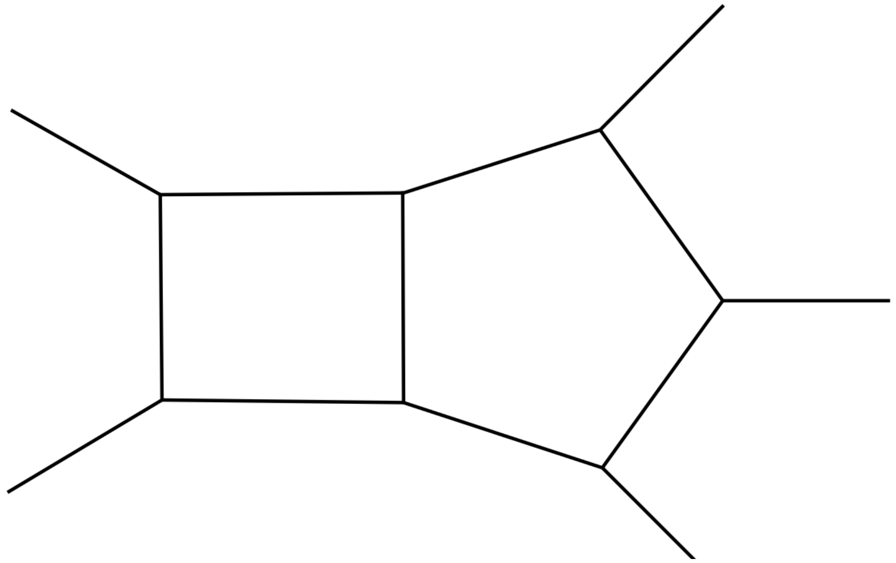
$$\mathcal{H} = \sum_{i,c} R_{i,c}\, \mathcal{I}_i\, C_c$$

direct evaluation




Feynman diagrams
→
helicity projection
→
colour decomposition
$\frac{s_{12}s_{13}+(d-4)s_{23}s_{14}}{s_{24}s_{35}}$
$\mathcal{H} = \sum_{i,c} R_{i,c}\, \mathcal{I}_i\, C_c$
direct evaluation

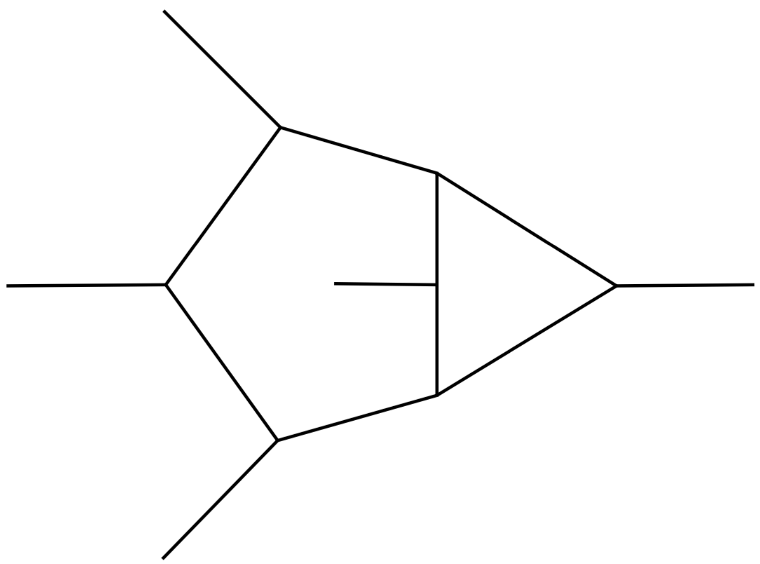

Feynman diagrams → helicity projection → colour decomposition
$\frac{s_{12}s_{13}+(d-4)s_{23}s_{14}}{s_{24}s_{35}}$
...
$\mathcal{H} = \sum_{i,c} R_{i,c}\, \mathcal{I}_i\, \mathcal{C}_c$
direct evaluation
scalar Feynman integrals



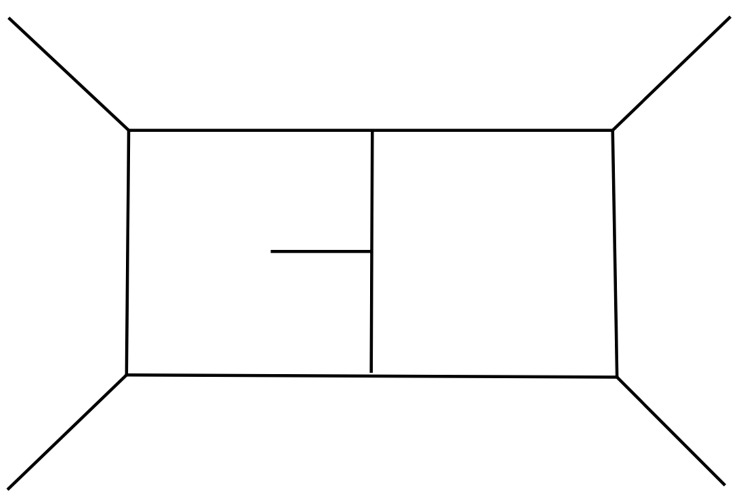
Papadopoulos, Tommasini, Wever: 1511.09404

Gehrmann, Henn, Lo Presti: 1511.05409

Gehrmann, Henn, Lo Presti: 1807.09812



Chicherin, Gehrmann, Henn, Wasser, Zhang et al: 1812.11160

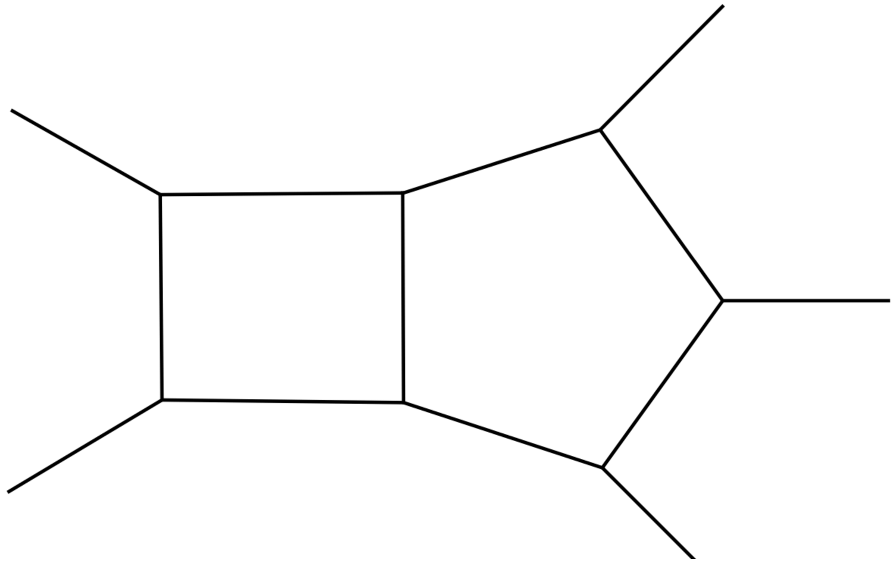


Chicherin, Gehrmann, Henn, Lo Presti, Mitev et al.: 1809.06240

Böhm, Georgoudis, Larsen, Schönemann, Zhang: 1805.01873

Abreu, Page, Zeng: 1807.11522

61



Papadopoulos, Tommasini, Wever: 1511.09404

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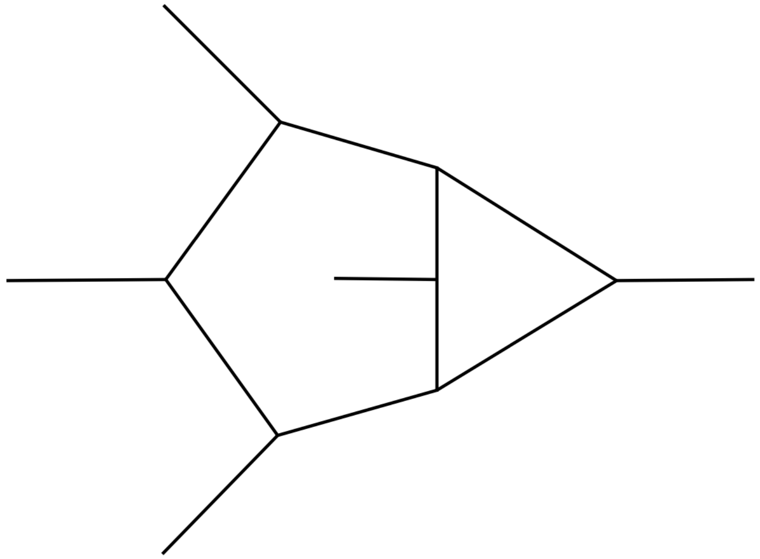
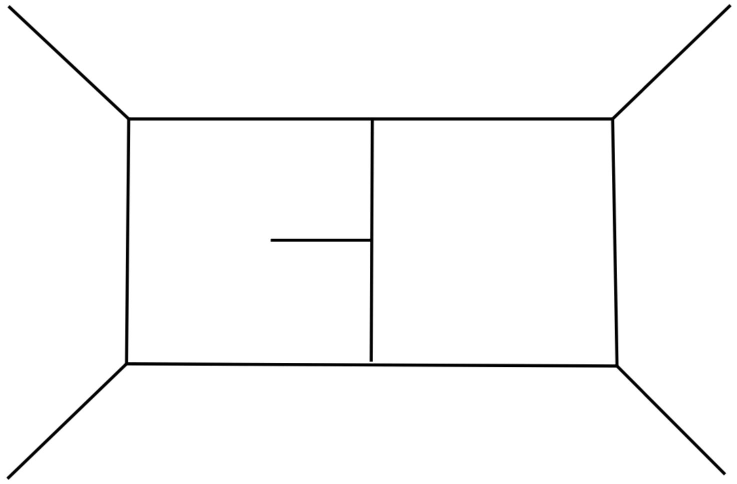
Gehrmann, Henn, Lo Presti: 1807.09812

73



Chicherin, Gehrmann, Henn, Wasser, Zhang et al: 1812.11160

108



Chicherin, Gehrmann, Henn, Lo Presti, Mitev et al.: 1809.06240

Böhm, Georgoudis, Larsen, Schönemann, Zhang: 1805.01873

Abreu, Page, Zeng: 1807.11522

Full canonical set available → Dmitry Chicherin, Vasily Sotnikov: 2009.07803

$$\mathcal{I} = \sum_{m \in masters} c_n(s_{ij}, d)\, \mathcal{M}_m$$

$$\mathcal{I} = \sum_{m \in masters} c_n(s_{ij}, d)\, \mathcal{M}_m$$

Finite-fields reconstruction

$$\mathcal{I} = \sum_{m \in masters} c_n(s_{ij}, d)\, \mathcal{M}_m$$

Finite-fields reconstruction

Denominator guessing

$$\mathcal{I} = \sum_{m \in masters} c_n(s_{ij}, d)\, \mathcal{M}_m$$

Finite-fields reconstruction

Denominator guessing

Syzygy techniques

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Finite-fields reconstruction

von Manteuffel, Schabinger: 1406.4513
Peraro: 1905.08019

Denominator guessing

Abreu, Dormans, Febres Cordero, Ita, Page: 1812.04586
Heller, von Manteuffel: 2101.08283

Syzygy techniques

Gluza, Kadja, Kosower: 1009.0472
Ita: 1510.05626
Larsen, Zhang: 1511.01071
Agarwal, Jones, von Manteuffel: 2011.15113

$$\mathcal{I} = \sum_{m \in masters} c_n(s_{ij}, d)\, \mathcal{M}_m$$

Finite-fields reconstruction

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$\longrightarrow$ FinRed

Multivariate Partial Fractioning

Multivariate Partial Fractioning

$$0 = 1 - d_0 s_{12}$$
$$0 = 1 - d_1 s_{23}$$
$$0 = 1 - d_2 s_{34}$$
$$0 = 1 - d_3 s_{45}$$
$$0 = 1 - d_4 s_{51}$$
$$0 = 1 - d_5(s_{12} + s_{23} - s_{45})$$
$$0 = 1 - d_6(s_{23} - s_{45} - s_{51})$$
$$0 = 1 - d_7(s_{23} + s_{34} - s_{51})$$
$$0 = 1 - d_8(s_{12} - s_{34} + s_{51})$$
$$0 = 1 - d_9(s_{12} - s_{34} - s_{45})$$
$$0 = 1 - d_{10}(s_{12} + s_{23})$$
$$0 = 1 - d_{11}(s_{12} - s_{34})$$
$$0 = 1 - d_{12}(s_{23} + s_{34})$$
$$0 = 1 - d_{13}(s_{12} - s_{45})$$
$$0 = 1 - d_{14}(s_{23} - s_{45})$$
$$0 = 1 - d_{15}(s_{12} + s_{23} - s_{34} - s_{45})$$
$$0 = 1 - d_{16}(s_{34} + s_{45})$$
$$0 = 1 - d_{17}(s_{23} - s_{51})$$
$$0 = 1 - d_{18}(s_{34} - s_{51})$$
$$0 = 1 - d_{19}(s_{12} + s_{23} - s_{45} - s_{51})$$
$$0 = 1 - d_{20}(s_{23} + s_{34} - s_{45} - s_{51})$$
$$0 = 1 - d_{21}(s_{12} + s_{51})$$
$$0 = 1 - d_{22}(s_{12} - s_{23} - s_{34} + s_{51})$$
$$0 = 1 - d_{23}(s_{12} - s_{34} - s_{45} + s_{51})$$
$$0 = 1 - d_{24}(s_{45} + s_{51})$$
$$0 = 1 - d_G(s_{23}^2 s_{12}^2 - 2s_{23}^2 s_{34} s_{12} + s_{23}^2 s_{34}^2 + 2s_{23}s_{34}s_{45}s_{12}$$
$$- 2s_{23}s_{34}^2 s_{45} + s_{34}^2 s_{45}^2 - 2s_{23}s_{51}s_{12}^2 + 2s_{23}s_{34}s_{51}s_{12} + 2s_{23}s_{45}s_{51}s_{12}$$
$$+ 2s_{34}s_{45}s_{51}s_{12} + 2s_{23}s_{34}s_{45}s_{51} - 2s_{34}s_{45}^2 s_{51} + s_{51}^2 s_{12}^2 - 2s_{45}s_{51}^2 s_{12} + s_{45}^2 s_{51}^2)$$

Multivariate Partial Fractioning

$$\langle 1 - d_1\, n_1, 1 - d_2\, n_2, \ldots \rangle$$

$$0 = 1 - d_0 s_{12}$$
$$0 = 1 - d_1 s_{23}$$
$$0 = 1 - d_2 s_{34}$$
$$0 = 1 - d_3 s_{45}$$
$$0 = 1 - d_4 s_{51}$$
$$0 = 1 - d_5 (s_{12} + s_{23} - s_{45})$$
$$0 = 1 - d_6 (s_{23} - s_{45} - s_{51})$$
$$0 = 1 - d_7 (s_{23} + s_{34} - s_{51})$$
$$0 = 1 - d_8 (s_{12} - s_{34} + s_{51})$$
$$0 = 1 - d_9 (s_{12} - s_{34} - s_{45})$$
$$0 = 1 - d_{10} (s_{12} + s_{23})$$
$$0 = 1 - d_{11} (s_{12} - s_{34})$$
$$0 = 1 - d_{12} (s_{23} + s_{34})$$
$$0 = 1 - d_{13} (s_{12} - s_{45})$$
$$0 = 1 - d_{14} (s_{23} - s_{45})$$
$$0 = 1 - d_{15} (s_{12} + s_{23} - s_{34} - s_{45})$$
$$0 = 1 - d_{16} (s_{34} + s_{45})$$
$$0 = 1 - d_{17} (s_{23} - s_{51})$$
$$0 = 1 - d_{18} (s_{34} - s_{51})$$
$$0 = 1 - d_{19} (s_{12} + s_{23} - s_{45} - s_{51})$$
$$0 = 1 - d_{20} (s_{23} + s_{34} - s_{45} - s_{51})$$
$$0 = 1 - d_{21} (s_{12} + s_{51})$$
$$0 = 1 - d_{22} (s_{12} - s_{23} - s_{34} + s_{51})$$
$$0 = 1 - d_{23} (s_{12} - s_{34} - s_{45} + s_{51})$$
$$0 = 1 - d_{24} (s_{45} + s_{51})$$
$$\begin{aligned} 0 = 1 - d_G (& s_{23}^2 s_{12}^2 - 2 s_{23}^2 s_{34} s_{12} + s_{23}^2 s_{34}^2 + 2 s_{23} s_{34} s_{45} s_{12} \\ & - 2 s_{23} s_{34}^2 s_{45} + s_{34}^2 s_{45}^2 - 2 s_{23} s_{51} s_{12}^2 + 2 s_{23} s_{34} s_{51} s_{12} + 2 s_{23} s_{45} s_{51} s_{12} \\ & + 2 s_{34} s_{45} s_{51} s_{12} + 2 s_{23} s_{34} s_{45} s_{51} - 2 s_{34} s_{45}^2 s_{51} + s_{51}^2 s_{12}^2 - 2 s_{45} s_{51}^2 s_{12} + s_{45}^2 s_{51}^2) \end{aligned}$$

Multivariate Partial Fractioning

$$\langle 1 - d_1\, n_1, 1 - d_2\, n_2, \ldots \rangle$$

$$\frac{1}{s_{23}} \frac{1}{(s_{23}+1)^2} \frac{1}{(s_{51}+1)} \frac{1}{(s_{23}-s_{45}-s_{51})}$$

$$0 = 1 - d_0 s_{12}$$
$$0 = 1 - d_1 s_{23}$$
$$0 = 1 - d_2 s_{34}$$
$$0 = 1 - d_3 s_{45}$$
$$0 = 1 - d_4 s_{51}$$
$$0 = 1 - d_5(s_{12} + s_{23} - s_{45})$$
$$0 = 1 - d_6(s_{23} - s_{45} - s_{51})$$
$$0 = 1 - d_7(s_{23} + s_{34} - s_{51})$$
$$0 = 1 - d_8(s_{12} - s_{34} + s_{51})$$
$$0 = 1 - d_9(s_{12} - s_{34} - s_{45})$$
$$0 = 1 - d_{10}(s_{12} + s_{23})$$
$$0 = 1 - d_{11}(s_{12} - s_{34})$$
$$0 = 1 - d_{12}(s_{23} + s_{34})$$
$$0 = 1 - d_{13}(s_{12} - s_{45})$$
$$0 = 1 - d_{14}(s_{23} - s_{45})$$
$$0 = 1 - d_{15}(s_{12} + s_{23} - s_{34} - s_{45})$$
$$0 = 1 - d_{16}(s_{34} + s_{45})$$
$$0 = 1 - d_{17}(s_{23} - s_{51})$$
$$0 = 1 - d_{18}(s_{34} - s_{51})$$
$$0 = 1 - d_{19}(s_{12} + s_{23} - s_{45} - s_{51})$$
$$0 = 1 - d_{20}(s_{23} + s_{34} - s_{45} - s_{51})$$
$$0 = 1 - d_{21}(s_{12} + s_{51})$$
$$0 = 1 - d_{22}(s_{12} - s_{23} - s_{34} + s_{51})$$
$$0 = 1 - d_{23}(s_{12} - s_{34} - s_{45} + s_{51})$$
$$0 = 1 - d_{24}(s_{45} + s_{51})$$
$$\begin{aligned}0 = 1 - d_G(&s_{23}^2 s_{12}^2 - 2s_{23}^2 s_{34} s_{12} + s_{23}^2 s_{34}^2 + 2s_{23}s_{34}s_{45}s_{12} \\ &- 2s_{23}s_{34}^2 s_{45} + s_{34}^2 s_{45}^2 - 2s_{23}s_{51}s_{12}^2 + 2s_{23}s_{34}s_{51}s_{12} + 2s_{23}s_{45}s_{51}s_{12} \\ &+ 2s_{34}s_{45}s_{51}s_{12} + 2s_{23}s_{34}s_{45}s_{51} - 2s_{34}s_{45}^2 s_{51} + s_{51}^2 s_{12}^2 - 2s_{45}s_{51}^2 s_{12} + s_{45}^2 s_{51}^2)\end{aligned}$$

Multivariate Partial Fractioning

$$0 = 1 - d_0 s_{12}$$
$$0 = 1 - d_1 s_{23}$$
$$0 = 1 - d_2 s_{34}$$
$$0 = 1 - d_3 s_{45}$$
$$0 = 1 - d_4 s_{51}$$
$$0 = 1 - d_5 (s_{12} + s_{23} - s_{45})$$
$$0 = 1 - d_6 (s_{23} - s_{45} - s_{51})$$
$$0 = 1 - d_7 (s_{23} + s_{34} - s_{51})$$
$$0 = 1 - d_8 (s_{12} - s_{34} + s_{51})$$
$$0 = 1 - d_9 (s_{12} - s_{34} - s_{45})$$
$$0 = 1 - d_{10} (s_{12} + s_{23})$$
$$0 = 1 - d_{11} (s_{12} - s_{34})$$
$$0 = 1 - d_{12} (s_{23} + s_{34})$$
$$0 = 1 - d_{13} (s_{12} - s_{45})$$
$$0 = 1 - d_{14} (s_{23} - s_{45})$$
$$0 = 1 - d_{15} (s_{12} + s_{23} - s_{34} - s_{45})$$
$$0 = 1 - d_{16} (s_{34} + s_{45})$$
$$0 = 1 - d_{17} (s_{23} - s_{51})$$
$$0 = 1 - d_{18} (s_{34} - s_{51})$$
$$0 = 1 - d_{19} (s_{12} + s_{23} - s_{45} - s_{51})$$
$$0 = 1 - d_{20} (s_{23} + s_{34} - s_{45} - s_{51})$$
$$0 = 1 - d_{21} (s_{12} + s_{51})$$
$$0 = 1 - d_{22} (s_{12} - s_{23} - s_{34} + s_{51})$$
$$0 = 1 - d_{23} (s_{12} - s_{34} - s_{45} + s_{51})$$
$$0 = 1 - d_{24} (s_{45} + s_{51})$$
$$0 = 1 - d_G (s_{23}^2 s_{12}^2 - 2 s_{23}^2 s_{34} s_{12} + s_{23}^2 s_{34}^2 + 2 s_{23} s_{34} s_{45} s_{12} - 2 s_{23} s_{34}^2 s_{45} + s_{34}^2 s_{45}^2 - 2 s_{23} s_{51} s_{12}^2 + 2 s_{23} s_{34} s_{51} s_{12} + 2 s_{23} s_{45} s_{51} s_{12} + 2 s_{34} s_{45} s_{51} s_{12} + 2 s_{23} s_{34} s_{45} s_{51} - 2 s_{34} s_{45}^2 s_{51} + s_{51}^2 s_{12}^2 - 2 s_{45} s_{51}^2 s_{12} + s_{45}^2 s_{51}^2)$$

$$\langle 1 - d_1\, n_1, 1 - d_2\, n_2, \ldots \rangle$$

$$\frac{1}{s_{23}} \frac{1}{(s_{23}+1)^2} \frac{1}{(s_{51}+1)} \frac{1}{(s_{23} - s_{45} - s_{51})} \rightarrow d_1\, d_6\, d_{10}^2\, d_{21}$$

Multivariate Partial Fractioning

$$0 = 1 - d_0 s_{12}$$
$$0 = 1 - d_1 s_{23}$$
$$0 = 1 - d_2 s_{34}$$
$$0 = 1 - d_3 s_{45}$$
$$0 = 1 - d_4 s_{51}$$
$$0 = 1 - d_5(s_{12} + s_{23} - s_{45})$$
$$0 = 1 - d_6(s_{23} - s_{45} - s_{51})$$
$$0 = 1 - d_7(s_{23} + s_{34} - s_{51})$$
$$0 = 1 - d_8(s_{12} - s_{34} + s_{51})$$
$$0 = 1 - d_9(s_{12} - s_{34} - s_{45})$$
$$0 = 1 - d_{10}(s_{12} + s_{23})$$
$$0 = 1 - d_{11}(s_{12} - s_{34})$$
$$0 = 1 - d_{12}(s_{23} + s_{34})$$
$$0 = 1 - d_{13}(s_{12} - s_{45})$$
$$0 = 1 - d_{14}(s_{23} - s_{45})$$
$$0 = 1 - d_{15}(s_{12} + s_{23} - s_{34} - s_{45})$$
$$0 = 1 - d_{16}(s_{34} + s_{45})$$
$$0 = 1 - d_{17}(s_{23} - s_{51})$$
$$0 = 1 - d_{18}(s_{34} - s_{51})$$
$$0 = 1 - d_{19}(s_{12} + s_{23} - s_{45} - s_{51})$$
$$0 = 1 - d_{20}(s_{23} + s_{34} - s_{45} - s_{51})$$
$$0 = 1 - d_{21}(s_{12} + s_{51})$$
$$0 = 1 - d_{22}(s_{12} - s_{23} - s_{34} + s_{51})$$
$$0 = 1 - d_{23}(s_{12} - s_{34} - s_{45} + s_{51})$$
$$0 = 1 - d_{24}(s_{45} + s_{51})$$
$$0 = 1 - d_G(s_{23}^2 s_{12}^2 - 2s_{23}^2 s_{34} s_{12} + s_{23}^2 s_{34}^2 + 2s_{23}s_{34}s_{45}s_{12} - 2s_{23}s_{34}^2 s_{45} + s_{34}^2 s_{45}^2 - 2s_{23}s_{51}s_{12}^2 + 2s_{23}s_{34}s_{51}s_{12} + 2s_{23}s_{45}s_{51}s_{12} + 2s_{34}s_{45}s_{51}s_{12} + 2s_{23}s_{34}s_{45}s_{51} - 2s_{34}s_{45}^2 s_{51} + s_{51}^2 s_{12}^2 - 2s_{45}s_{51}^2 s_{12} + s_{45}^2 s_{51}^2)$$

$$\langle 1 - d_1\, n_1, 1 - d_2\, n_2, \ldots \rangle$$

$$\frac{1}{s_{23}} \frac{1}{(s_{23}+1)^2} \frac{1}{(s_{51}+1)} \frac{1}{(s_{23} - s_{45} - s_{51})} \to d_1\, d_6\, d_{10}^2\, d_{21}$$

$$= -d_5 d_6 d_3^2 + d_6 d_{10} d_3^2 - d_5 d_{21} d_3^2 + d_{10} d_{21} d_3^2 + d_6 d_{10}^2 d_3 - d_5 d_6 d_3 + d_6 d_{10} d_3 + d_{10}^2 d_{21} d_3 - d_5 d_{21} d_3 + d_{10} d_{21} d_3 + d_1 d_5 d_6 + d_1 d_5 d_{21}$$

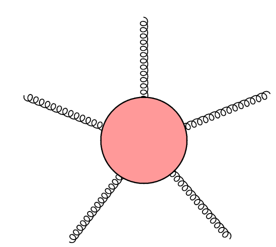
Multivariate Partial Fractioning

$$0 = 1 - d_0 s_{12}$$
$$0 = 1 - d_1 s_{23}$$
$$0 = 1 - d_2 s_{34}$$
$$0 = 1 - d_3 s_{45}$$
$$0 = 1 - d_4 s_{51}$$
$$0 = 1 - d_5(s_{12} + s_{23} - s_{45})$$
$$0 = 1 - d_6(s_{23} - s_{45} - s_{51})$$
$$0 = 1 - d_7(s_{23} + s_{34} - s_{51})$$
$$0 = 1 - d_8(s_{12} - s_{34} + s_{51})$$
$$0 = 1 - d_9(s_{12} - s_{34} - s_{45})$$
$$0 = 1 - d_{10}(s_{12} + s_{23})$$
$$0 = 1 - d_{11}(s_{12} - s_{34})$$
$$0 = 1 - d_{12}(s_{23} + s_{34})$$
$$0 = 1 - d_{13}(s_{12} - s_{45})$$
$$0 = 1 - d_{14}(s_{23} - s_{45})$$
$$0 = 1 - d_{15}(s_{12} + s_{23} - s_{34} - s_{45})$$
$$0 = 1 - d_{16}(s_{34} + s_{45})$$
$$0 = 1 - d_{17}(s_{23} - s_{51})$$
$$0 = 1 - d_{18}(s_{34} - s_{51})$$
$$0 = 1 - d_{19}(s_{12} + s_{23} - s_{45} - s_{51})$$
$$0 = 1 - d_{20}(s_{23} + s_{34} - s_{45} - s_{51})$$
$$0 = 1 - d_{21}(s_{12} + s_{51})$$
$$0 = 1 - d_{22}(s_{12} - s_{23} - s_{34} + s_{51})$$
$$0 = 1 - d_{23}(s_{12} - s_{34} - s_{45} + s_{51})$$
$$0 = 1 - d_{24}(s_{45} + s_{51})$$
$$0 = 1 - d_G(s_{23}^2 s_{12}^2 - 2s_{23}^2 s_{34} s_{12} + s_{23}^2 s_{34}^2 + 2s_{23}s_{34}s_{45}s_{12} - 2s_{23}s_{34}^2 s_{45} + s_{34}^2 s_{45}^2 - 2s_{23}s_{51}s_{12}^2 + 2s_{23}s_{34}s_{51}s_{12} + 2s_{23}s_{45}s_{51}s_{12} + 2s_{34}s_{45}s_{51}s_{12} + 2s_{23}s_{34}s_{45}s_{51} - 2s_{34}s_{45}^2 s_{51} + s_{51}^2 s_{12}^2 - 2s_{45}s_{51}^2 s_{12} + s_{45}^2 s_{51}^2)$$

$$\langle 1 - d_1\, n_1, 1 - d_2\, n_2, \ldots \rangle$$

$$\frac{1}{s_{23}} \frac{1}{(s_{23}+1)^2} \frac{1}{(s_{51}+1)} \frac{1}{(s_{23}-s_{45}-s_{51})} \to d_1\, d_6\, d_{10}^2\, d_{21}$$

$$= -d_5 d_6 d_3^2 + d_6 d_{10} d_3^2 - d_5 d_{21} d_3^2 + d_{10} d_{21} d_3^2 + d_6 d_{10}^2 d_3 - d_5 d_6 d_3 + d_6 d_{10} d_3 + d_{10}^2 d_{21} d_3 - d_5 d_{21} d_3 + d_{10} d_{21} d_3 + d_1 d_5 d_6 + d_1 d_5 d_{21}$$

The largest IBP for 

Multivariate Partial Fractioning

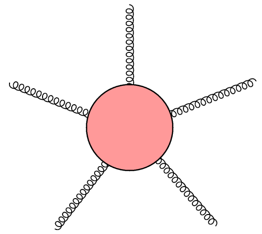
$0 = 1 - d_0 s_{12}$
$0 = 1 - d_1 s_{23}$
$0 = 1 - d_2 s_{34}$
$0 = 1 - d_3 s_{45}$
$0 = 1 - d_4 s_{51}$
$0 = 1 - d_5(s_{12} + s_{23} - s_{45})$
$0 = 1 - d_6(s_{23} - s_{45} - s_{51})$
$0 = 1 - d_7(s_{23} + s_{34} - s_{51})$
$0 = 1 - d_8(s_{12} - s_{34} + s_{51})$
$0 = 1 - d_9(s_{12} - s_{34} - s_{45})$
$0 = 1 - d_{10}(s_{12} + s_{23})$
$0 = 1 - d_{11}(s_{12} - s_{34})$
$0 = 1 - d_{12}(s_{23} + s_{34})$
$0 = 1 - d_{13}(s_{12} - s_{45})$
$0 = 1 - d_{14}(s_{23} - s_{45})$
$0 = 1 - d_{15}(s_{12} + s_{23} - s_{34} - s_{45})$
$0 = 1 - d_{16}(s_{34} + s_{45})$
$0 = 1 - d_{17}(s_{23} - s_{51})$
$0 = 1 - d_{18}(s_{34} - s_{51})$
$0 = 1 - d_{19}(s_{12} + s_{23} - s_{45} - s_{51})$
$0 = 1 - d_{20}(s_{23} + s_{34} - s_{45} - s_{51})$
$0 = 1 - d_{21}(s_{12} + s_{51})$
$0 = 1 - d_{22}(s_{12} - s_{23} - s_{34} + s_{51})$
$0 = 1 - d_{23}(s_{12} - s_{34} - s_{45} + s_{51})$
$0 = 1 - d_{24}(s_{45} + s_{51})$

$$0 = 1 - d_G(s_{23}^2 s_{12}^2 - 2s_{23}^2 s_{34} s_{12} + s_{23}^2 s_{34}^2 + 2s_{23}s_{34}s_{45}s_{12} - 2s_{23}s_{34}^2 s_{45} + s_{34}^2 s_{45}^2 - 2s_{23}s_{51}s_{12}^2 + 2s_{23}s_{34}s_{51}s_{12} + 2s_{23}s_{45}s_{51}s_{12} + 2s_{34}s_{45}s_{51}s_{12} + 2s_{23}s_{34}s_{45}s_{51} - 2s_{34}s_{45}^2 s_{51} + s_{51}^2 s_{12}^2 - 2s_{45}s_{51}^2 s_{12} + s_{45}^2 s_{51}^2)$$

$$\langle 1 - d_1\, n_1, 1 - d_2\, n_2, \ldots \rangle$$

$$\frac{1}{s_{23}} \frac{1}{(s_{23}+1)^2} \frac{1}{(s_{51}+1)} \frac{1}{(s_{23}-s_{45}-s_{51})} \to d_1\, d_6\, d_{10}^2\, d_{21}$$

$$= -d_5 d_6 d_3^2 + d_6 d_{10} d_3^2 - d_5 d_{21} d_3^2 + d_{10} d_{21} d_3^2 + d_6 d_{10}^2 d_3 - d_5 d_6 d_3 + d_6 d_{10} d_3 + d_{10}^2 d_{21} d_3 - d_5 d_{21} d_3 + d_{10} d_{21} d_3 + d_1 d_5 d_6 + d_1 d_5 d_{21}$$

The largest IBP for 

Common
Denominator
3GB

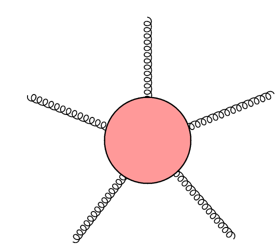
Multivariate Partial Fractioning

$$0=1-d_0 s_{12}$$
$$0=1-d_1 s_{23}$$
$$0=1-d_2 s_{34}$$
$$0=1-d_3 s_{45}$$
$$0=1-d_4 s_{51}$$
$$0=1-d_5(s_{12}+s_{23}-s_{45})$$
$$0=1-d_6(s_{23}-s_{45}-s_{51})$$
$$0=1-d_7(s_{23}+s_{34}-s_{51})$$
$$0=1-d_8(s_{12}-s_{34}+s_{51})$$
$$0=1-d_9(s_{12}-s_{34}-s_{45})$$
$$0=1-d_{10}(s_{12}+s_{23})$$
$$0=1-d_{11}(s_{12}-s_{34})$$
$$0=1-d_{12}(s_{23}+s_{34})$$
$$0=1-d_{13}(s_{12}-s_{45})$$
$$0=1-d_{14}(s_{23}-s_{45})$$
$$0=1-d_{15}(s_{12}+s_{23}-s_{34}-s_{45})$$
$$0=1-d_{16}(s_{34}+s_{45})$$
$$0=1-d_{17}(s_{23}-s_{51})$$
$$0=1-d_{18}(s_{34}-s_{51})$$
$$0=1-d_{19}(s_{12}+s_{23}-s_{45}-s_{51})$$
$$0=1-d_{20}(s_{23}+s_{34}-s_{45}-s_{51})$$
$$0=1-d_{21}(s_{12}+s_{51})$$
$$0=1-d_{22}(s_{12}-s_{23}-s_{34}+s_{51})$$
$$0=1-d_{23}(s_{12}-s_{34}-s_{45}+s_{51})$$
$$0=1-d_{24}(s_{45}+s_{51})$$
$$0=1-d_G(s_{23}^2s_{12}^2-2s_{23}^2s_{34}s_{12}+s_{23}^2s_{34}^2+2s_{23}s_{34}s_{45}s_{12} -2s_{23}s_{34}^2s_{45}+s_{34}^2s_{45}^2-2s_{23}s_{51}s_{12}^2+2s_{23}s_{34}s_{51}s_{12}+2s_{23}s_{45}s_{51}s_{12} +2s_{34}s_{45}s_{51}s_{12}+2s_{23}s_{34}s_{45}s_{51}-2s_{34}s_{45}^2s_{51}+s_{51}^2s_{12}^2-2s_{45}s_{51}^2s_{12}+s_{45}^2s_{51}^2)$$

$$\langle 1-d_1\,n_1, 1-d_2\,n_2,\ldots\rangle$$

$$\frac{1}{s_{23}}\frac{1}{(s_{23}+1)^2}\frac{1}{(s_{51}+1)}\frac{1}{(s_{23}-s_{45}-s_{51})}\to d_1\,d_6\,d_{10}^2\,d_{21}$$

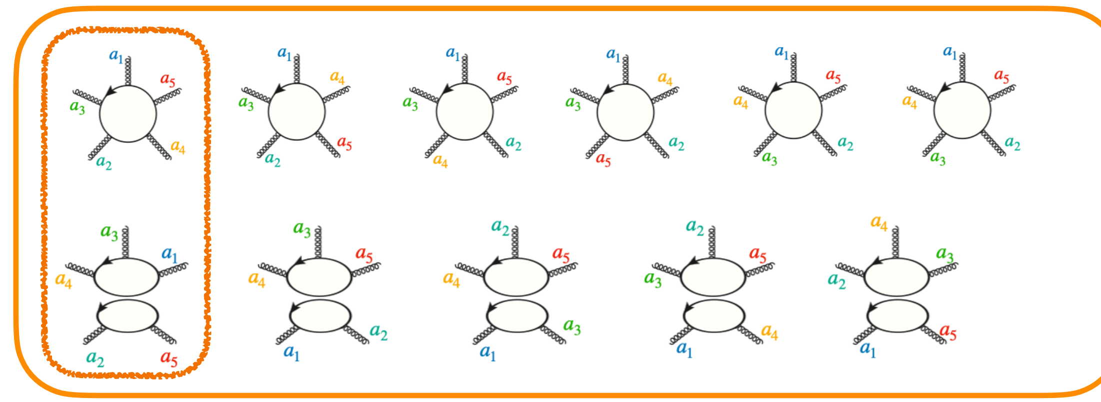
$$= -d_5d_6d_3^2+d_6d_{10}d_3^2-d_5d_{21}d_3^2+d_{10}d_{21}d_3^2+ d_6d_{10}^2d_3-d_5d_6d_3+d_6d_{10}d_3+d_{10}^2d_{21}d_3- d_5d_{21}d_3+d_{10}d_{21}d_3+d_1d_5d_6+d_1d_5d_{21}$$

The largest IBP for 

Common Denominator 3GB —Singular→ Partial Fractioned 24MB

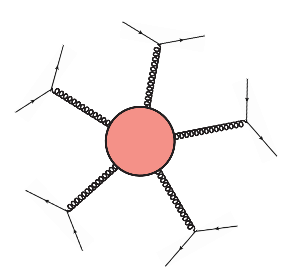
Feynman diagrams → helicity projection → colour decomposition

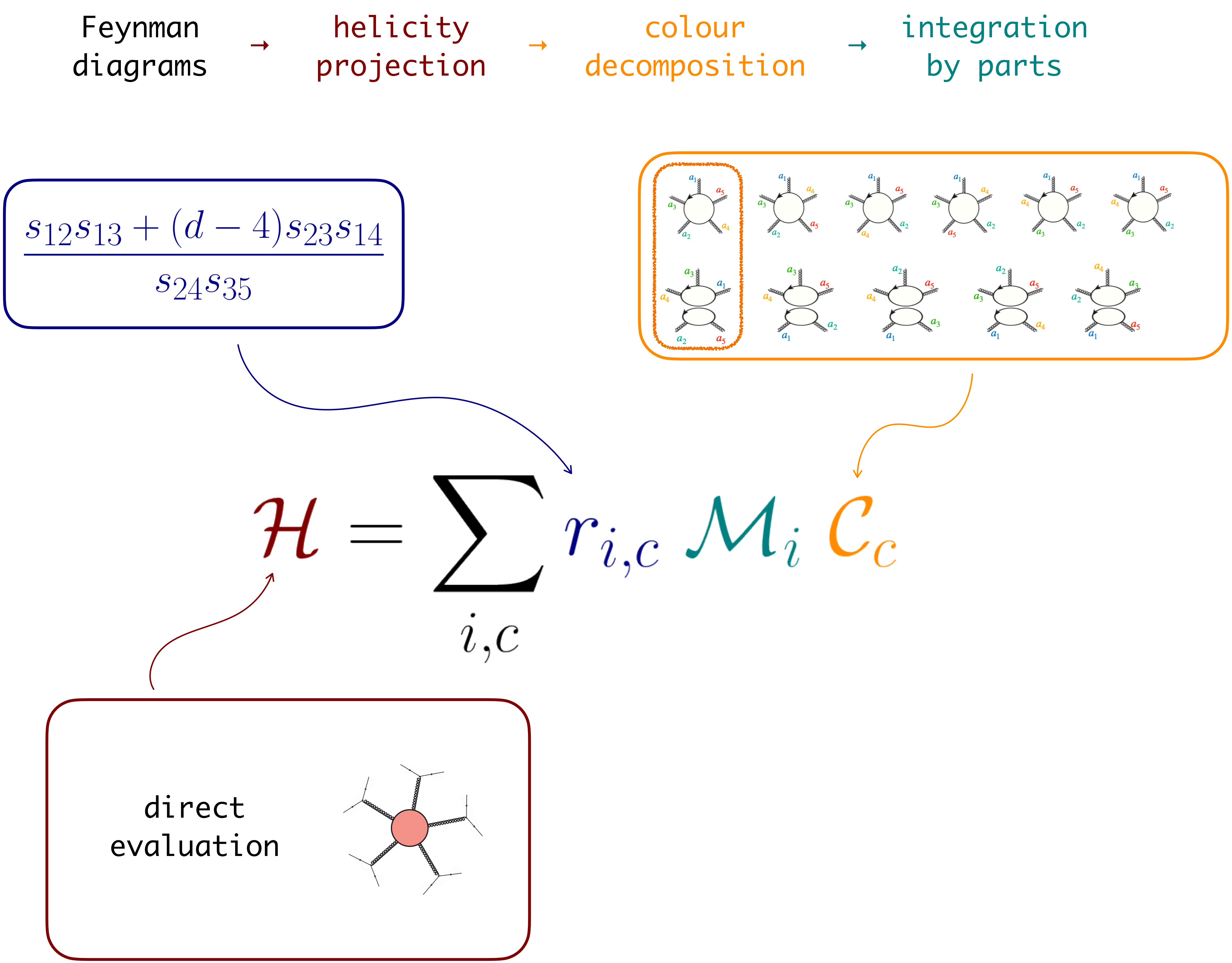
$$\frac{s_{12}s_{13} + (d-4)s_{23}s_{14}}{s_{24}s_{35}}$$

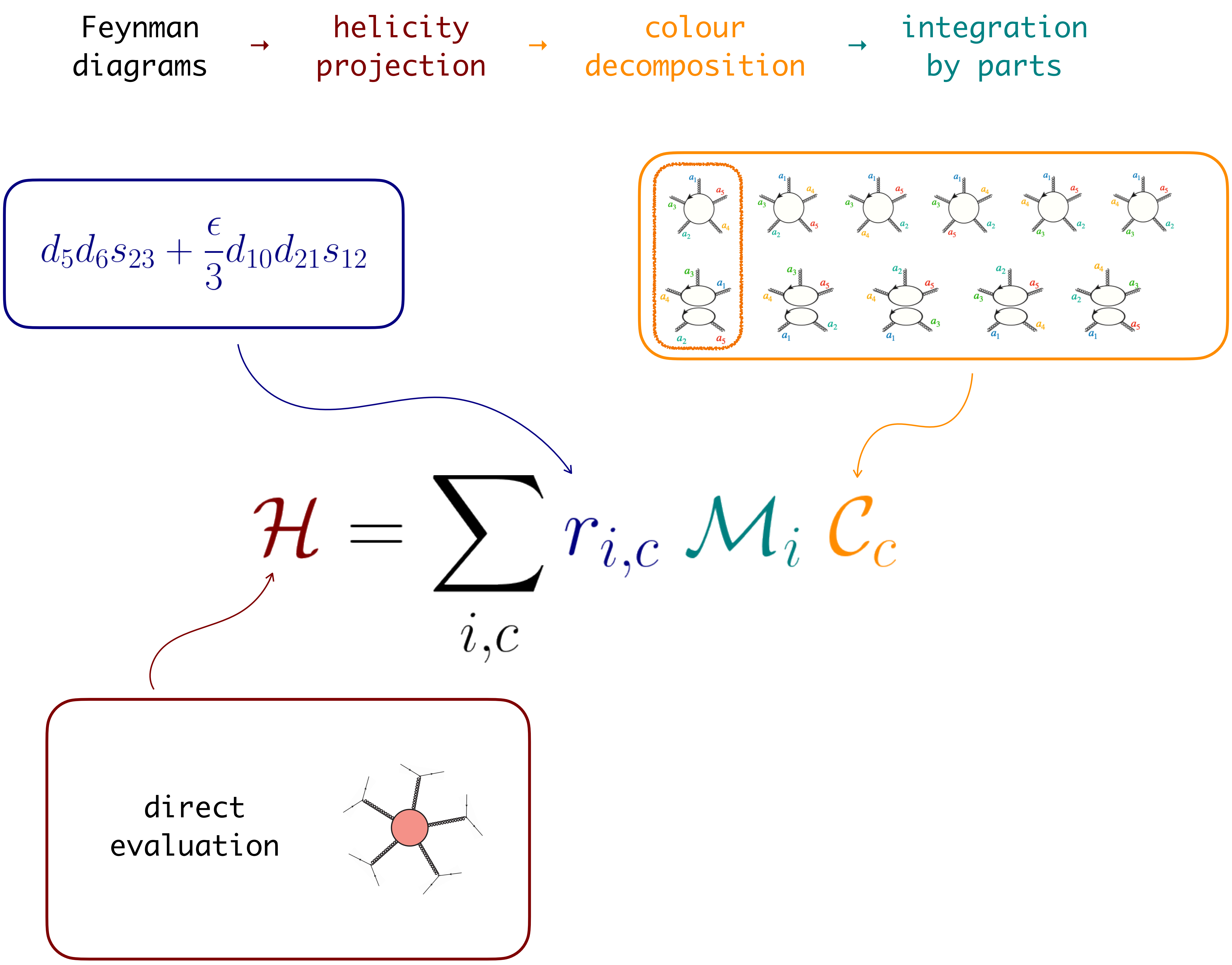


$$\mathcal{H} = \sum_{i,c} R_{i,c}\, \mathcal{I}_i\, C_c$$

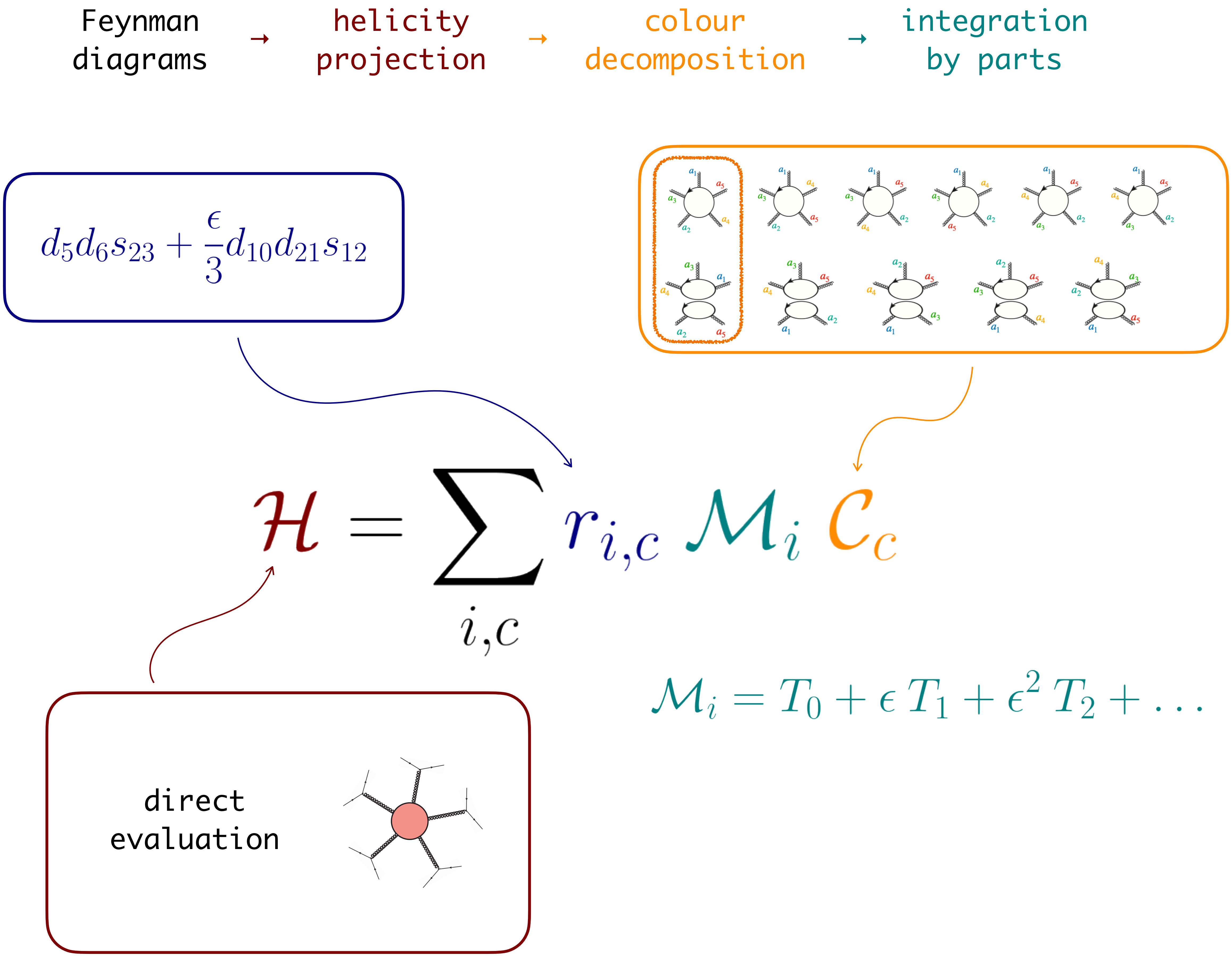
direct evaluation




Feynman diagrams → helicity projection → colour decomposition → integration by parts
$$\frac{s_{12}s_{13}+(d-4)s_{23}s_{14}}{s_{24}s_{35}}$$
$$\mathcal{H}=\sum_{i,c} r_{i,c}\,\mathcal{M}_i\,\mathcal{C}_c$$
direct evaluation

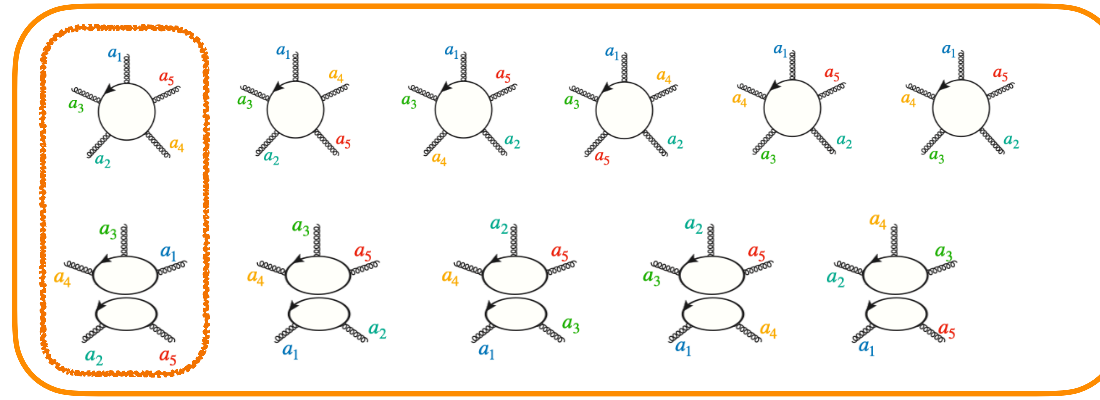


Feynman diagrams → helicity projection → colour decomposition → integration by parts
$d_5 d_6 s_{23} + \frac{\epsilon}{3} d_{10} d_{21} s_{12}$
$\mathcal{H} = \sum_{i,c} r_{i,c} \, \mathcal{M}_i \, \mathcal{C}_c$
direct evaluation


Feynman diagrams → helicity projection → colour decomposition → integration by parts
$d_5 d_6 s_{23} + \frac{\epsilon}{3} d_{10} d_{21} s_{12}$
$\mathcal{H} = \sum_{i,c} r_{i,c}\, \mathcal{M}_i\, \mathcal{C}_c$
$\mathcal{M}_i = T_0 + \epsilon\, T_1 + \epsilon^2\, T_2 + \ldots$
direct evaluation

Feynman diagrams → helicity projection → colour decomposition → integration by parts

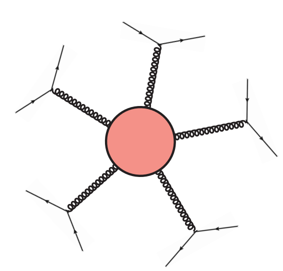
$$d_5 d_6 s_{23} + \frac{\epsilon}{3} d_{10} d_{21} s_{12}$$



$$\mathcal{H} = \sum_{i,c} r_{i,c}\, \mathcal{M}_i\, \mathcal{C}_c$$

$$\mathcal{M}_i = T_0 + \epsilon\, T_1 + \epsilon^2\, T_2 + \ldots$$

direct evaluation

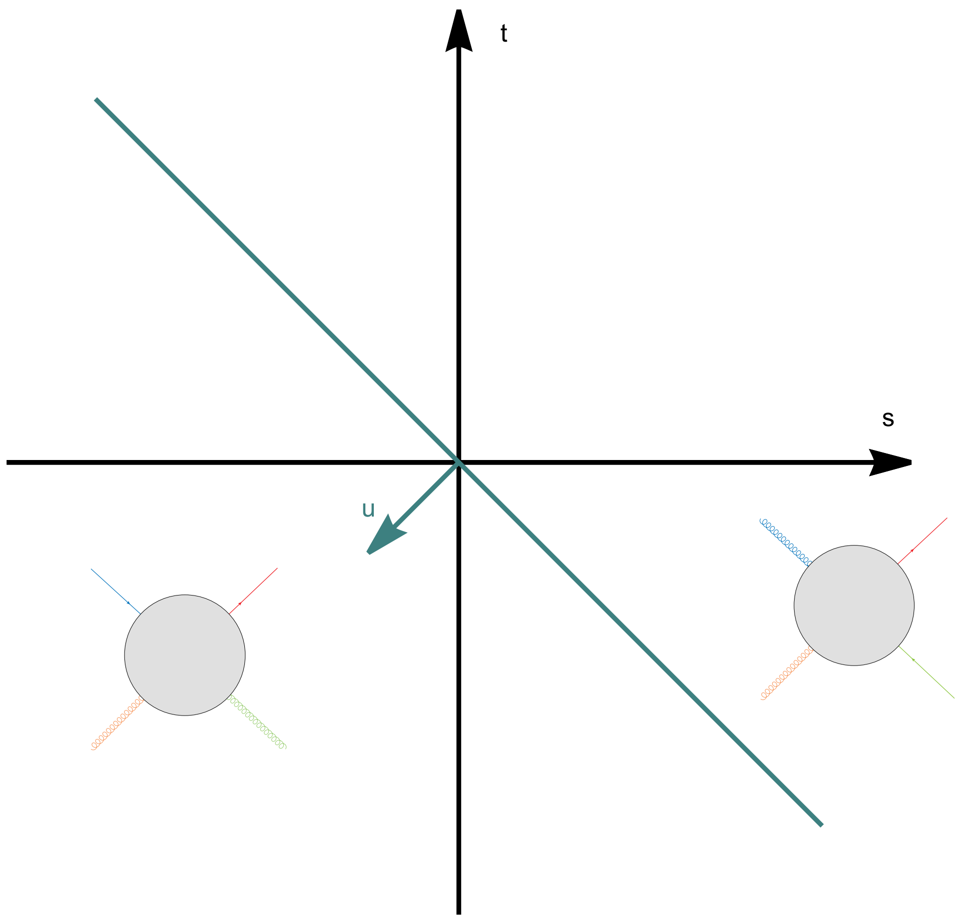


ϵ-expansion

Analytic Continuation

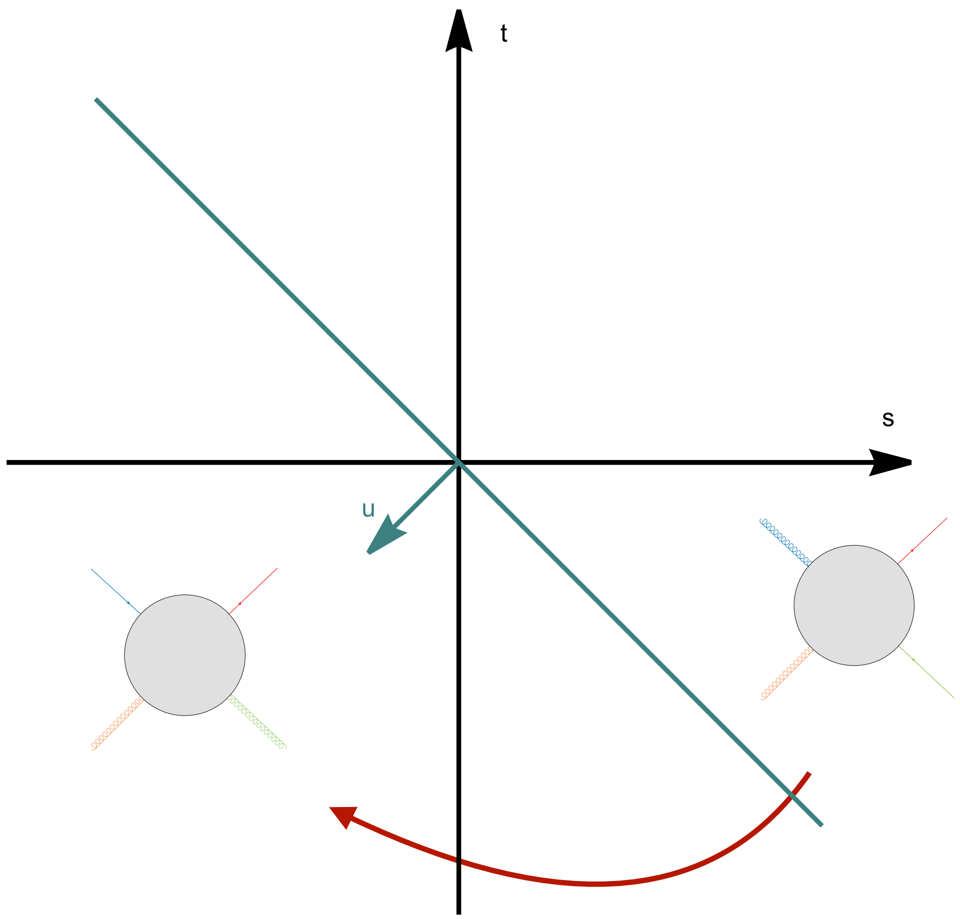
Analytic Continuation

At 4 points…



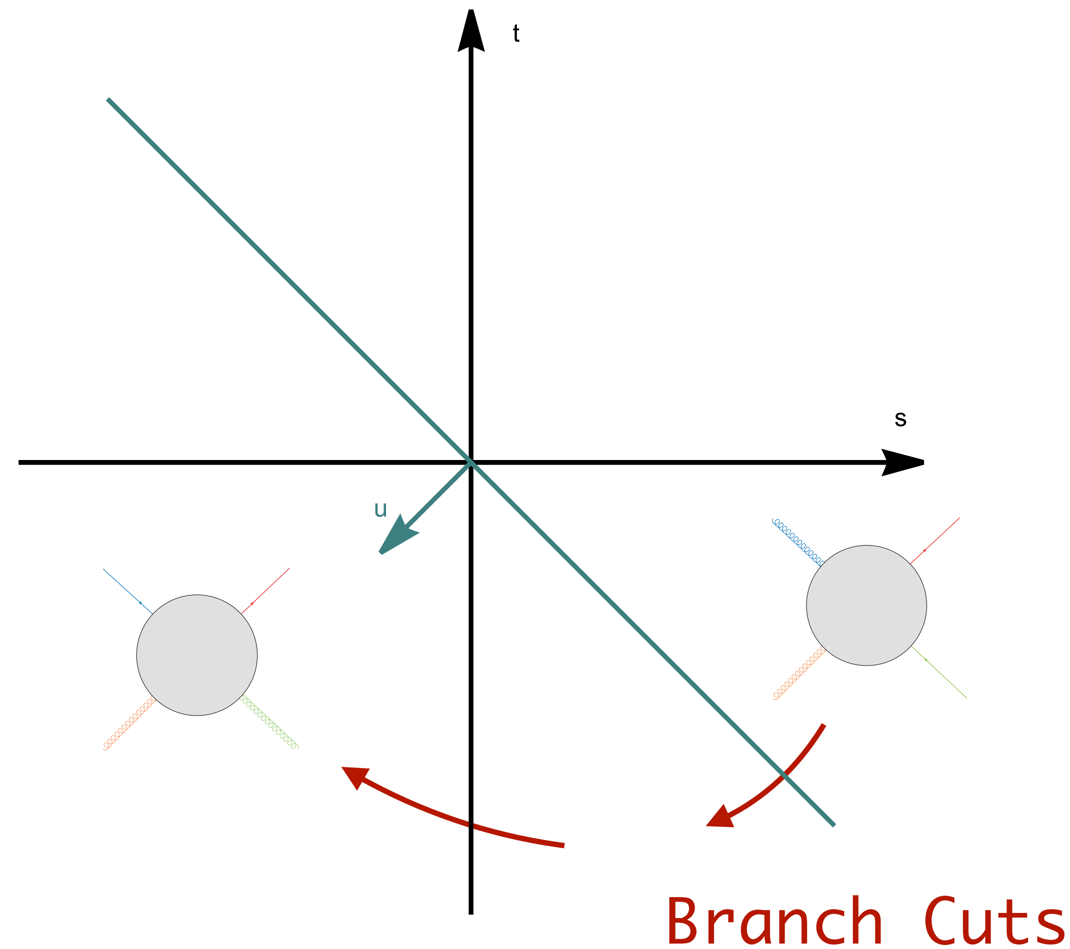
Analytic Continuation

At 4 points…



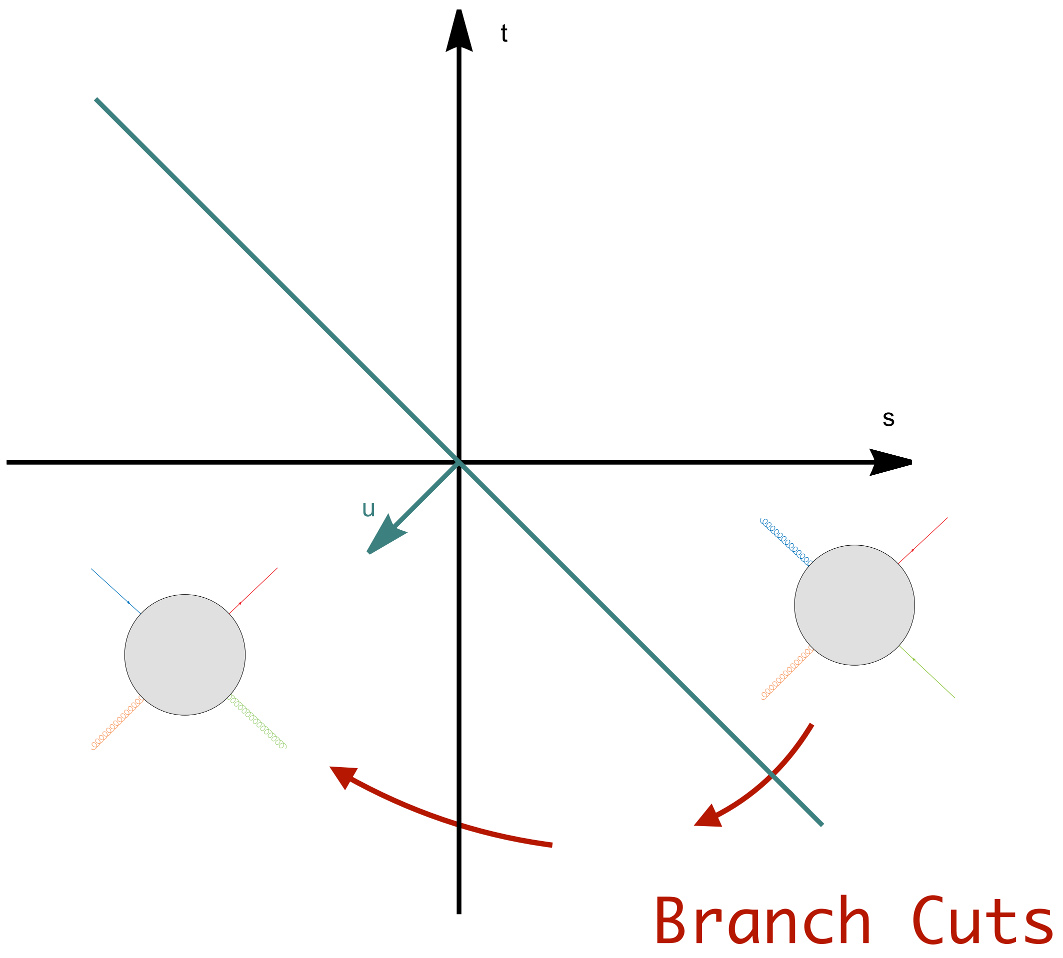
Analytic Continuation

At 4 points…



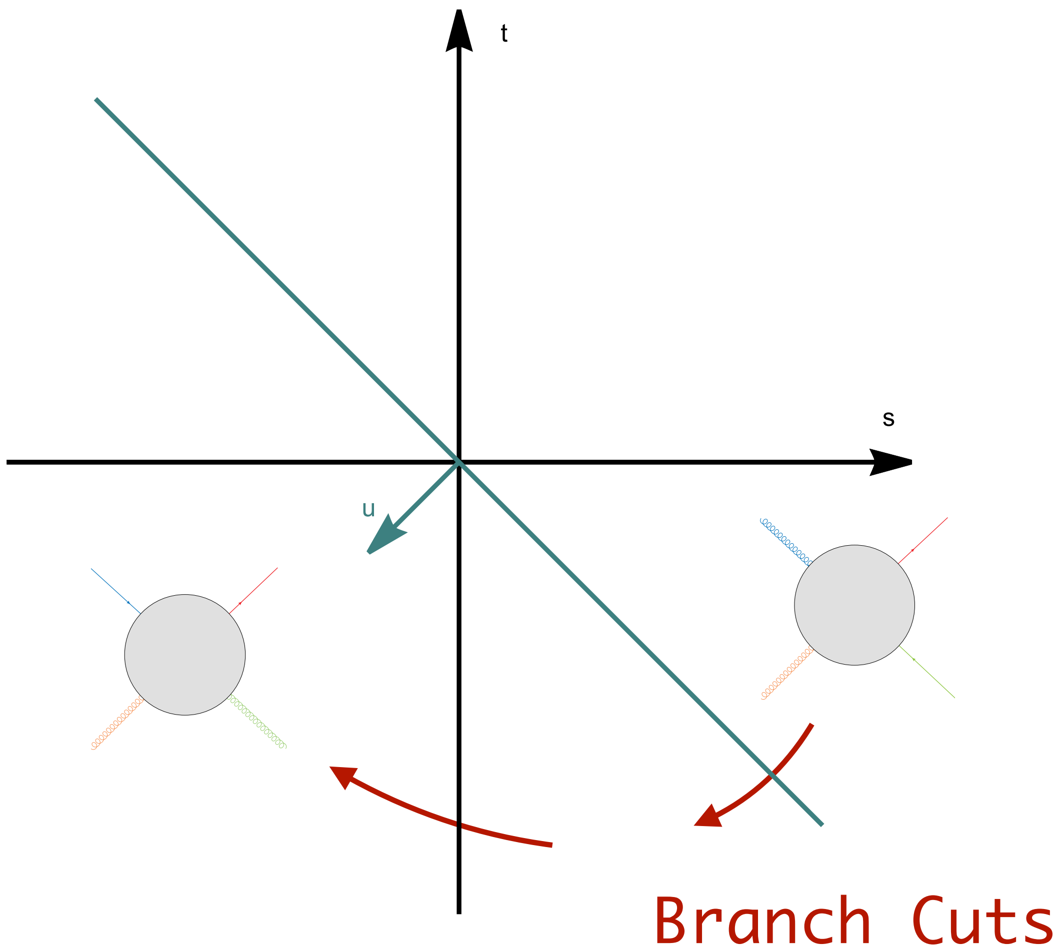
Analytic Continuation

At 4 points…



At 5 points…

Analytic Continuation



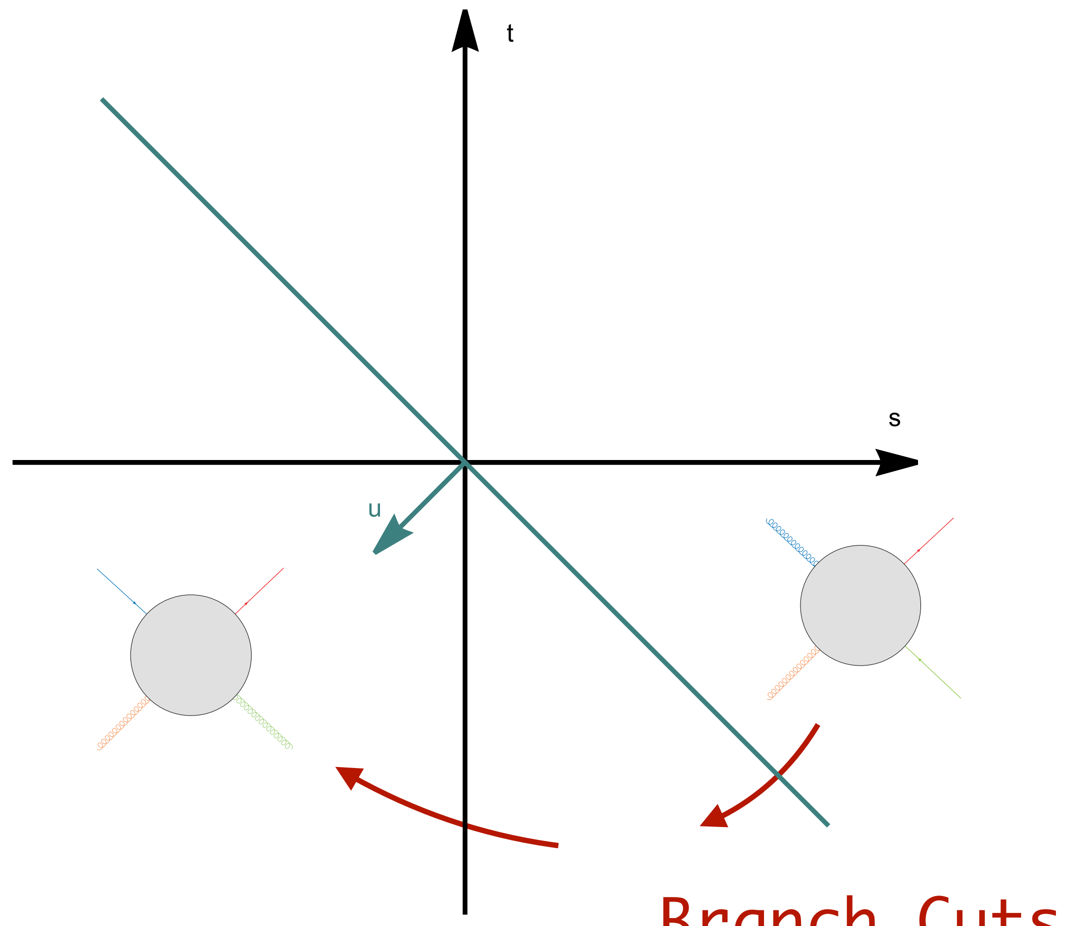
At 4 points…

At 5 points…

$$\sigma[\mathcal{M}(p_i)] = \mathcal{M}(\sigma[p_i])$$

Analytic Continuation

At 4 points…



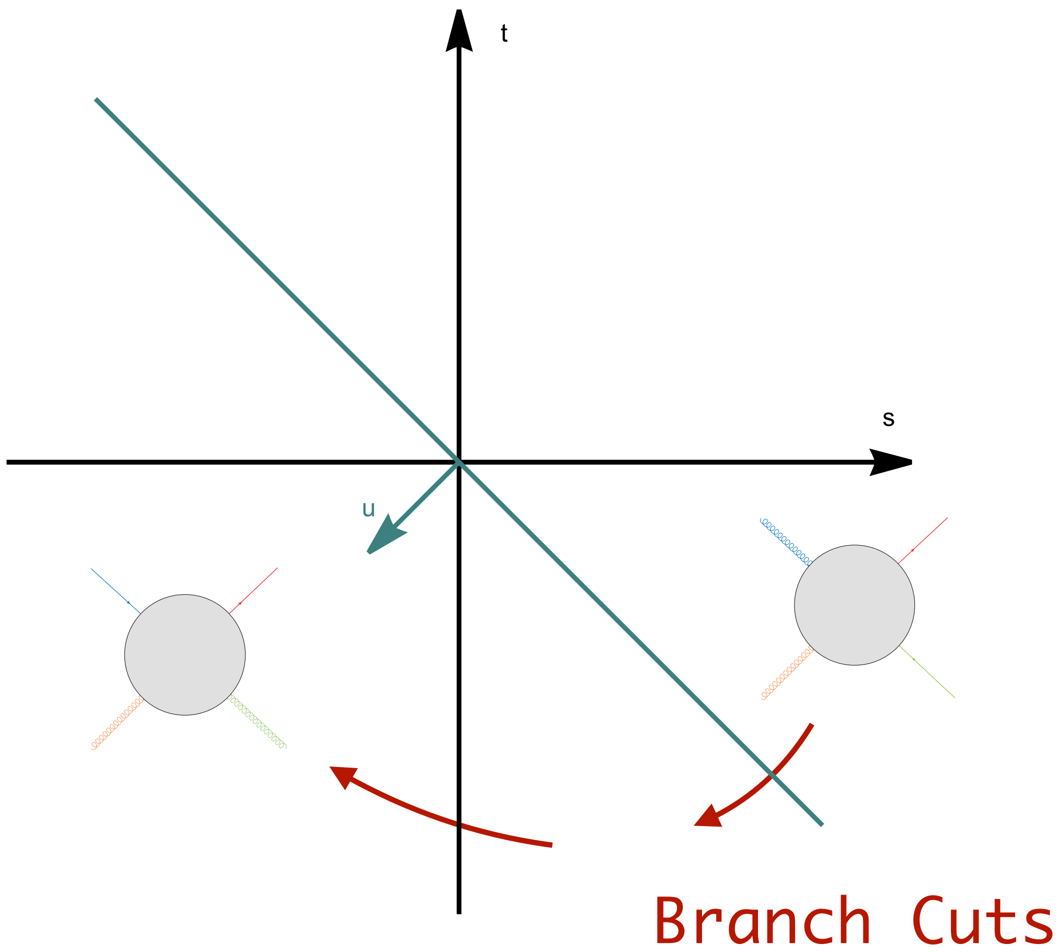
At 5 points…

$$\sigma = \{1 \to 2, 2 \to 3, 3 \to 4, 4 \to 5, 5 \to 1\}$$

$$\sigma[\mathcal{M}(p_i)] = \mathcal{M}(\sigma[p_i])$$

Analytic Continuation

At 4 points…



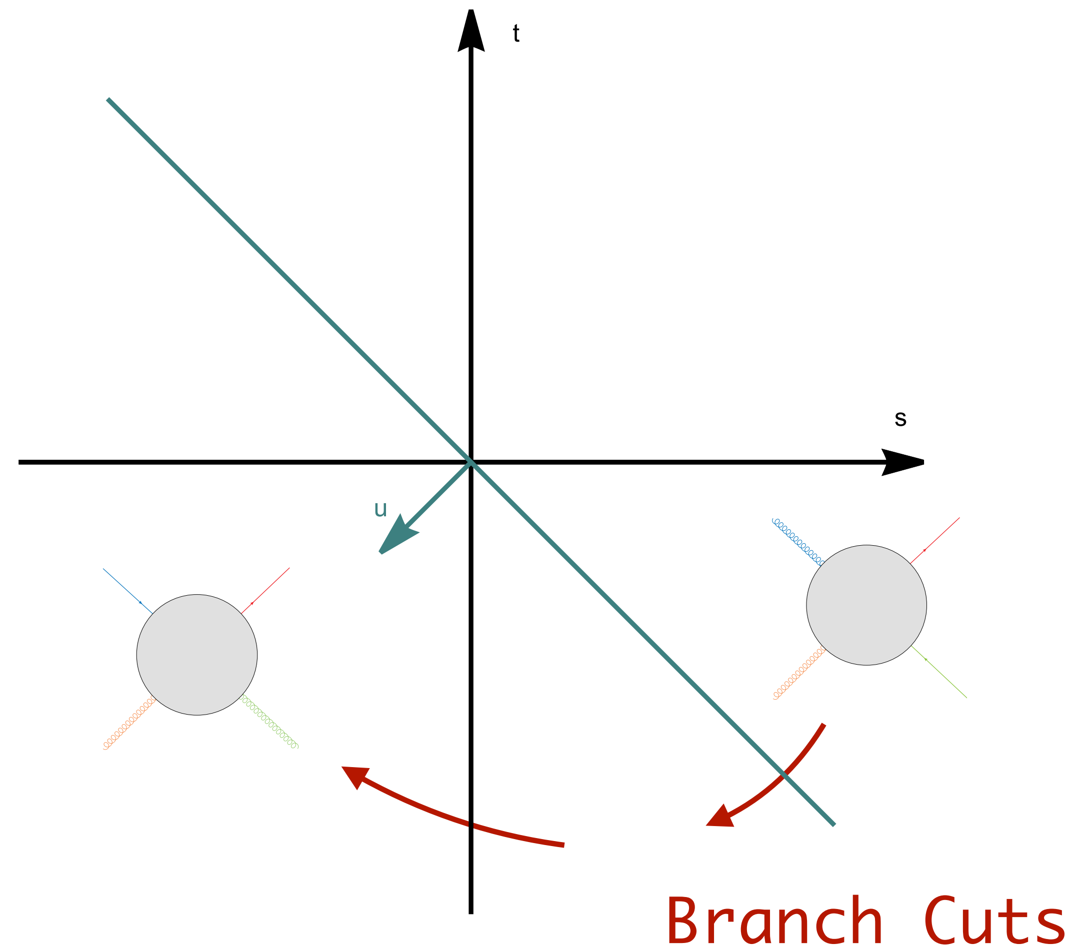
At 5 points…

$$\sigma = \{1 \to 2, 2 \to 3, 3 \to 4, 4 \to 5, 5 \to 1\}$$

$$\sigma[\mathcal{M}(p_i)] = \mathcal{M}(\sigma[p_i]) = \mathcal{M}'(p_i)$$

Analytic Continuation

At 4 points…



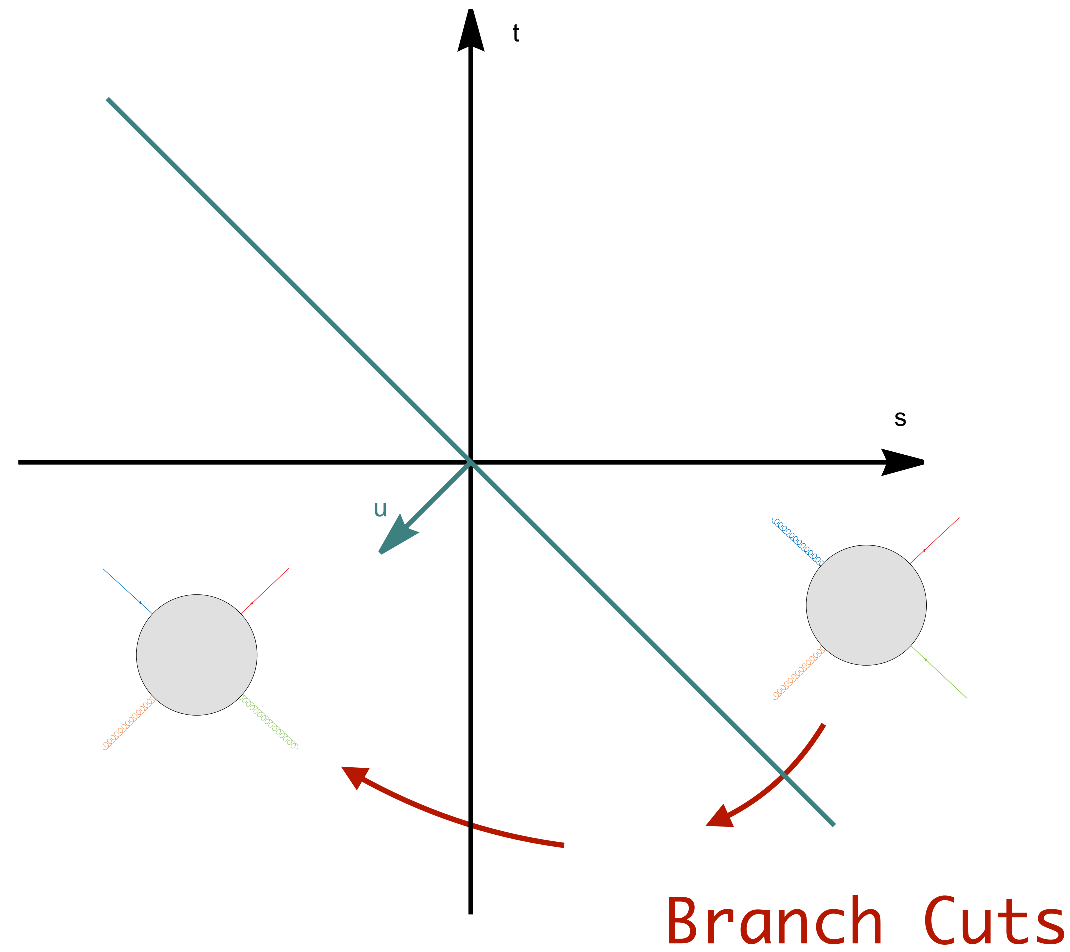
At 5 points… $\sigma = \{1 \to 2, 2 \to 3, 3 \to 4, 4 \to 5, 5 \to 1\}$

$$\sigma[\mathcal{M}(p_i)] = \mathcal{M}(\sigma[p_i]) = \mathcal{M}'(p_i)$$

$$= T_0' + \epsilon\, T_1' + \epsilon^2\, T_2' + \ldots$$

Analytic Continuation

At 4 points…



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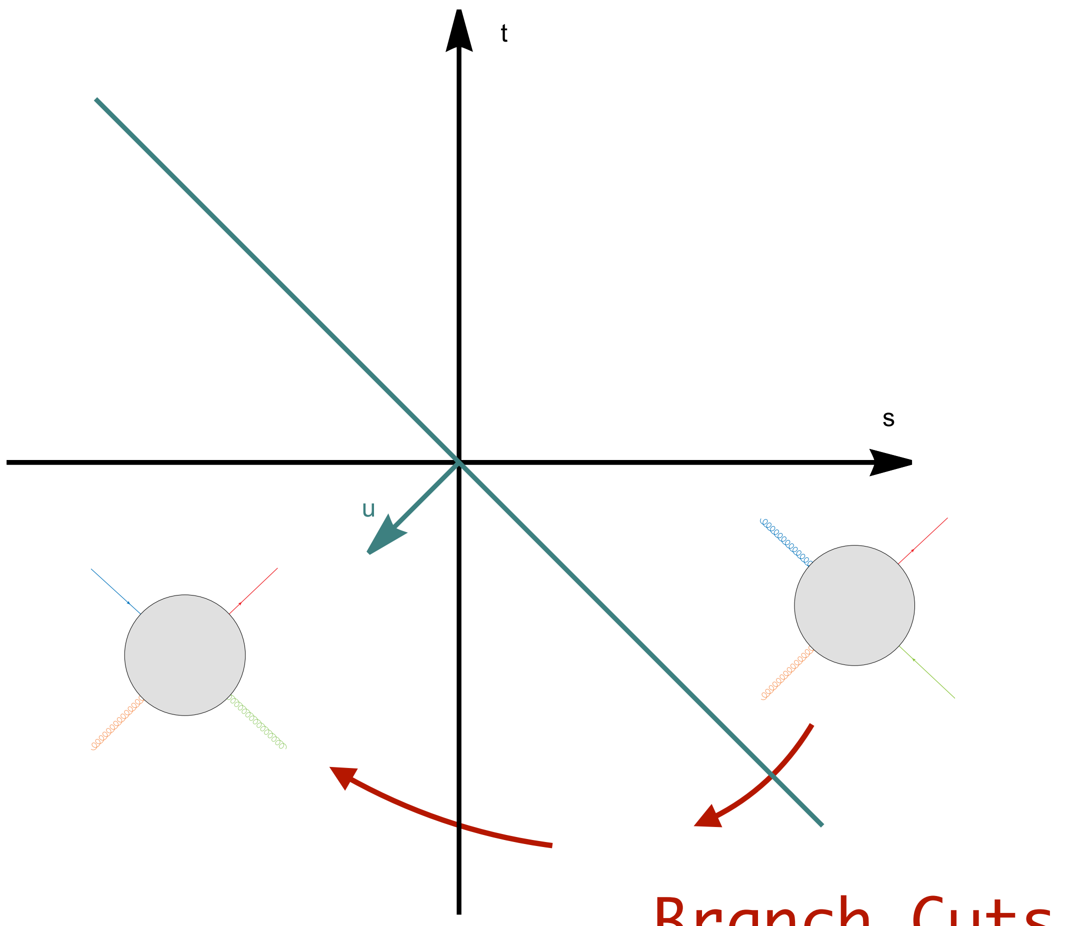
$$\sigma[\mathcal{M}(p_i)] = \mathcal{M}(\sigma[p_i]) = \mathcal{M}'(p_i)$$

$$\sigma[T_0 + \epsilon\, T_1 + \epsilon^2\, T_2 + \ldots] = T_0' + \epsilon\, T_1' + \epsilon^2\, T_2' + \ldots$$

Analytic Continuation

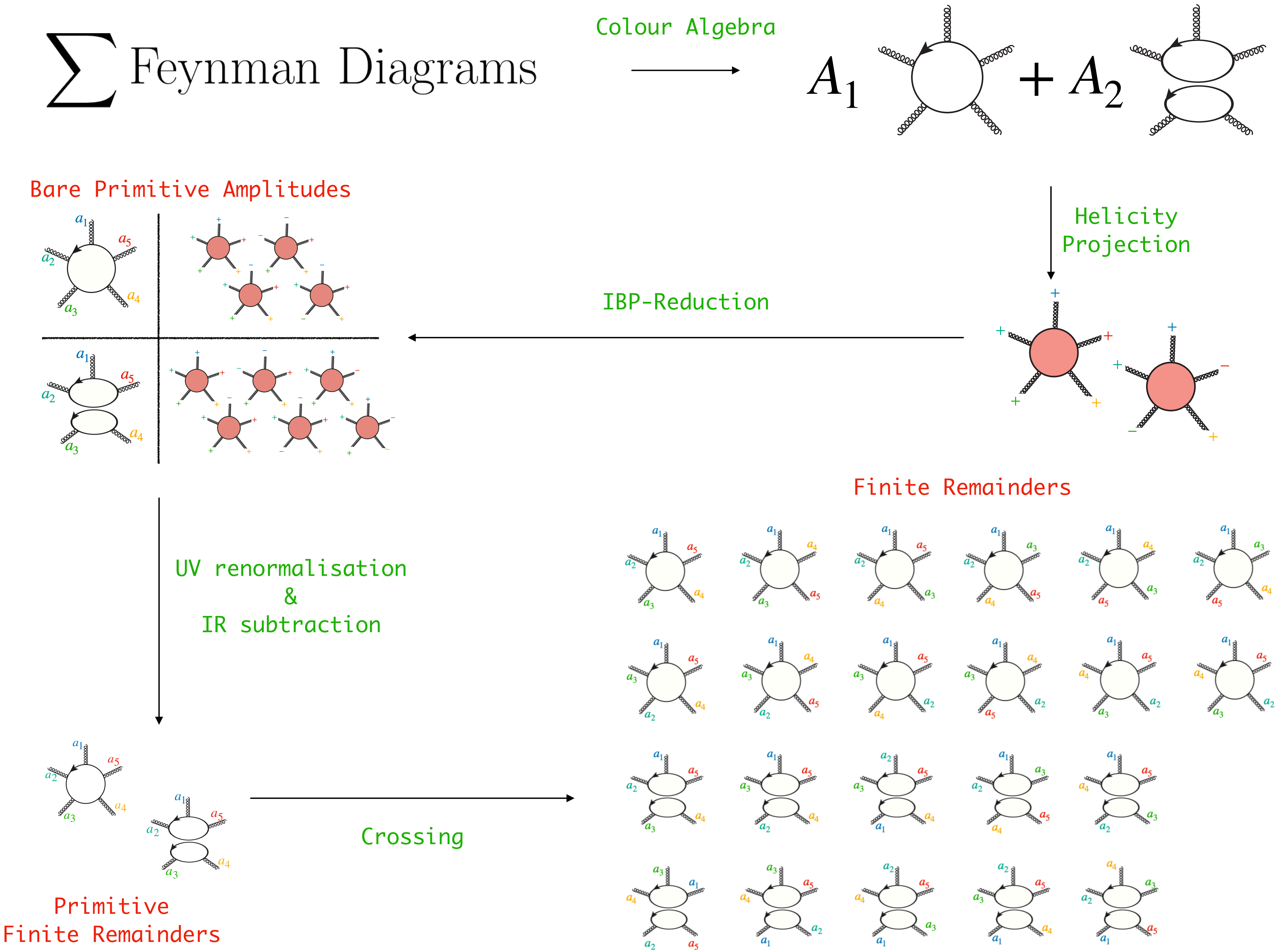
At 4 points…



At 5 points… $\sigma = \{1 \to 2, 2 \to 3, 3 \to 4, 4 \to 5, 5 \to 1\}$

$$\sigma[\mathcal{M}(p_i)] = \mathcal{M}(\sigma[p_i]) = \mathcal{M}'(p_i)$$

$$\sigma[T_0] + \epsilon\,\sigma[T_1] + \epsilon^2\,\sigma[T_2] + \ldots = T_0' + \epsilon\,T_1' + \epsilon^2\,T_2' + \ldots$$



$\sum$ Feynman Diagrams
Colour Algebra
A_1 + A_2
Helicity Projection
IBP-Reduction
Bare Primitive Amplitudes
UV renormalisation & IR subtraction
Finite Remainders
Crossing
Primitive Finite Remainders

Constraints

Constraints

- UV renormalisation

Constraints

- UV renormalisation
- IR factorisation

Catani: 9802439
Gardi, Magnea: 0901.1091, 0908.3273
Becher, Neubert: 0903.1126

Constraints

- UV renormalisation
- IR factorisation

$$\mathcal{H}_{renorm.} = \mathcal{Z}\,\mathcal{H}_{finite}$$

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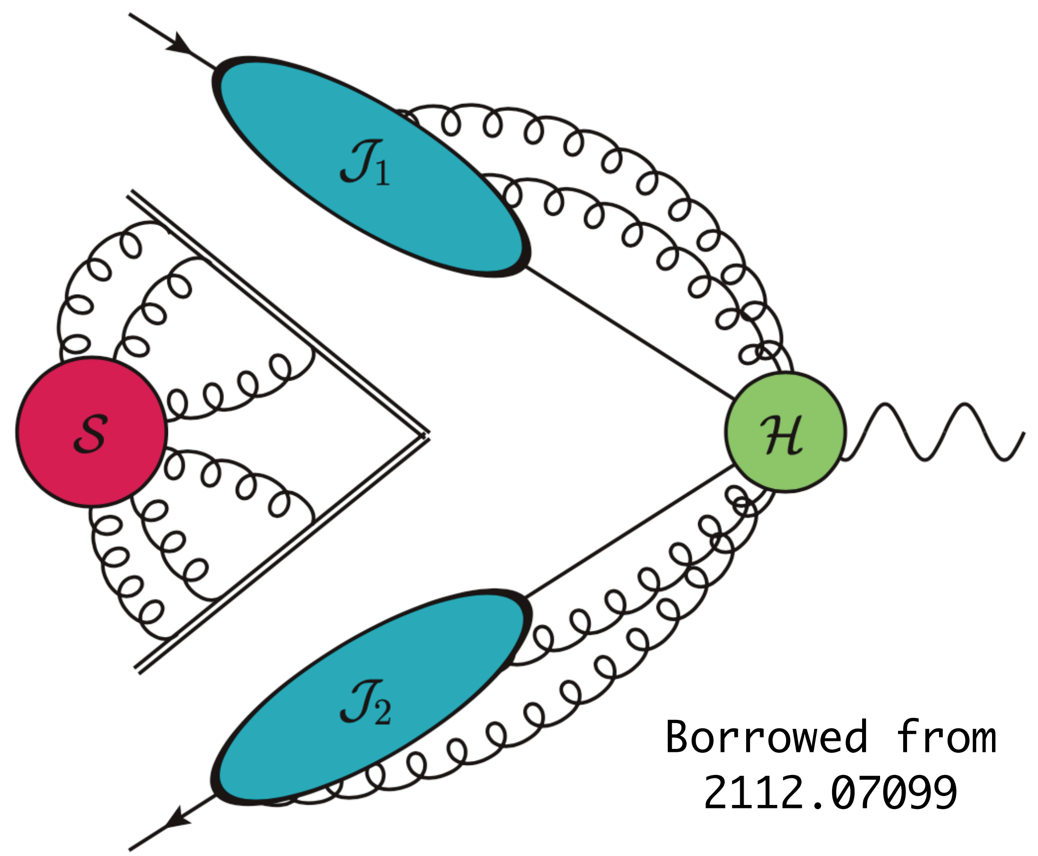

Borrowed from
2112.07099

Catani: 9802439
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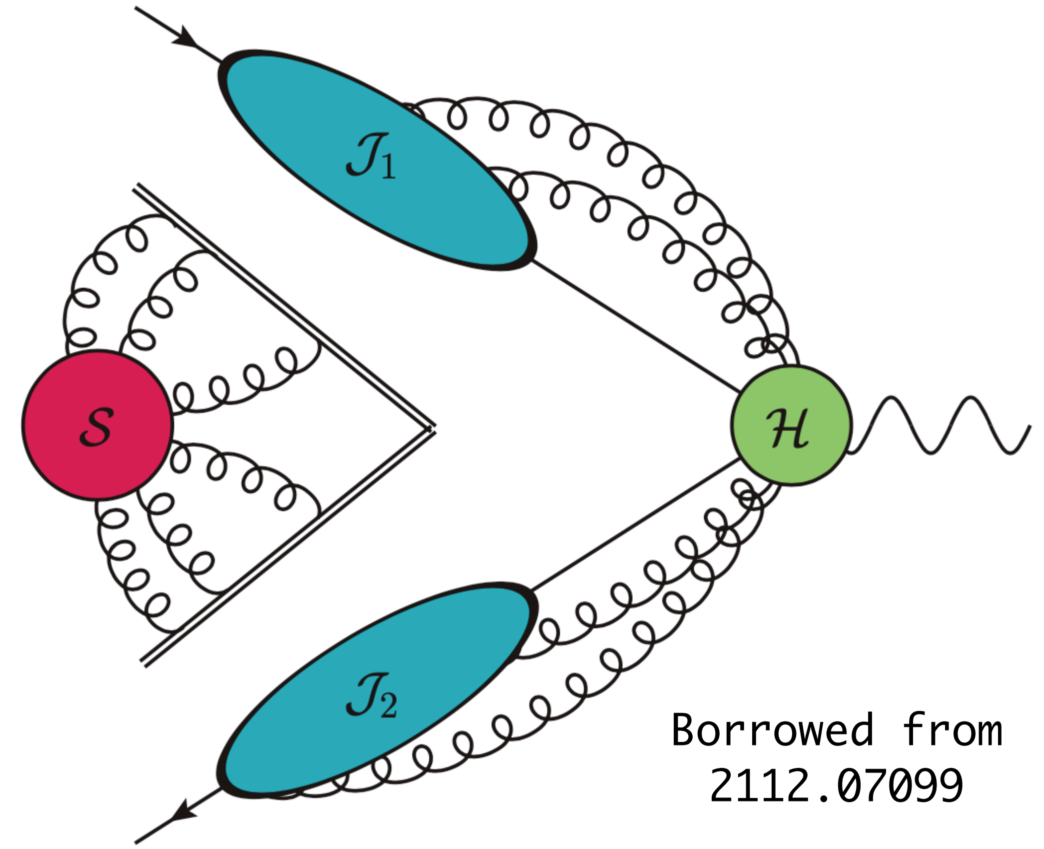

Borrowed from 2112.07099

$$\mathcal{H}_{renorm.} = \mathcal{Z}\,\mathcal{H}_{finite}$$

$$\boldsymbol{\mathcal{Z}}^{-1} = \mathbf{I} - \left(\frac{\alpha_s}{2\pi}\right)\boldsymbol{\mathcal{I}}_1 - \left(\frac{\alpha_s}{2\pi}\right)^2\boldsymbol{\mathcal{I}}_2 + \mathrm{O}(\alpha_s^3)$$

Catani: 9802439
Gardi, Magnea: 0901.1091, 0908.3273
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Z. Bern, L. Dixon, D. A. Kosower: hep-ph/9302280

S. Abreu, F. Febres Cordero, H. Ita,
B. Page, V. Sotnikov: 2102.13609

S. Badger, D. Chicherin, T. Gehrmann, G. Heinrich,
J.M. Henn, T. Peraro, P. Wasser, Y. Zhang,
S. Zoia: 1905.03733

Constraints

- UV renormalisation ✓
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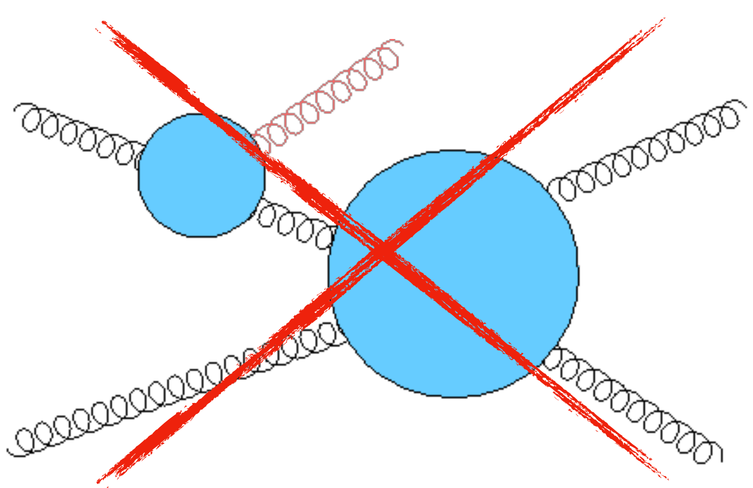

- 2 loop leading colour ✓
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full colour pp → jjj ✓

full colour pp → jjj ✓

input for N^{2}LO cross section

a step towards pp → jj at N^{3}LO

full colour pp → jjj ✓

input for N^2LO cross section

a step towards pp → jj at N^3LO

managing the complexity is possible

full colour pp → jjj ✓

input for N^2LO cross section

a step towards pp → jj at N^3LO

managing the complexity is possible

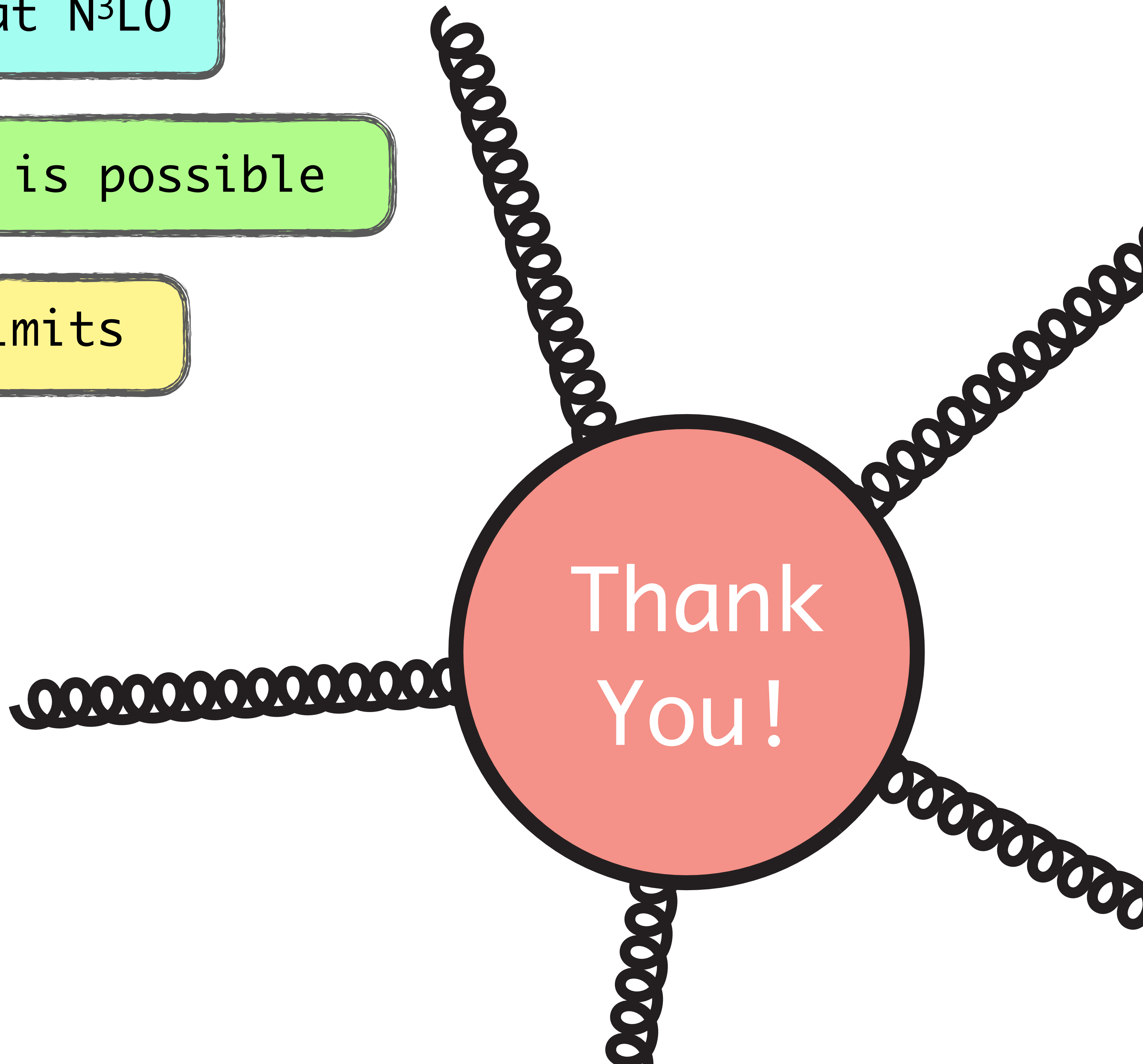
study/confirm special limits

full colour $pp \to jjj$ ✓

input for N^2LO cross section
a step towards $pp \to jj$ at N^3LO

managing the complexity is possible

study/confirm special limits



Backup

How to manage the complexity

How to manage the complexity

Numerically

Analytically

How to manage the complexity

Numerically

- faster
- manageable complexity

Analytically

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- manageable complexity

- subtraction of infinities
- has to be repeated

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- once and for all
- full information
- infinities are regulated

How to manage the complexity

Numerically

- faster
- manageable complexity

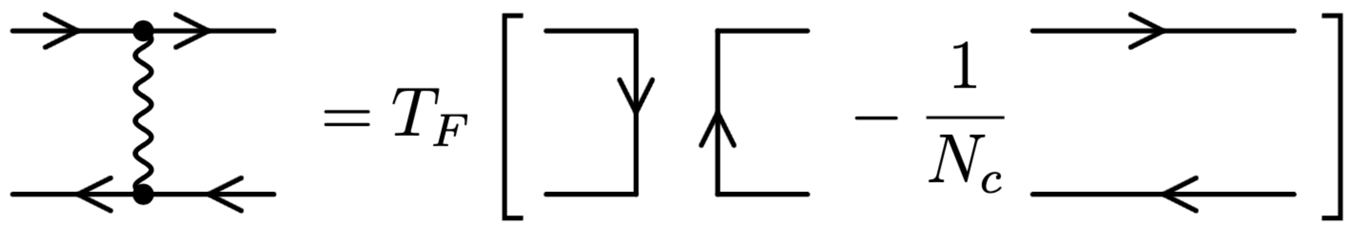
- subtraction of infinities
- has to be repeated

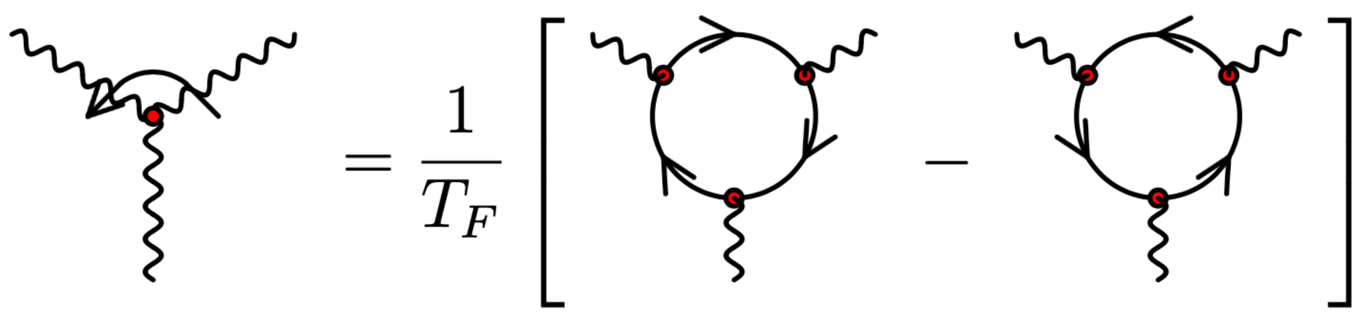
Analytically

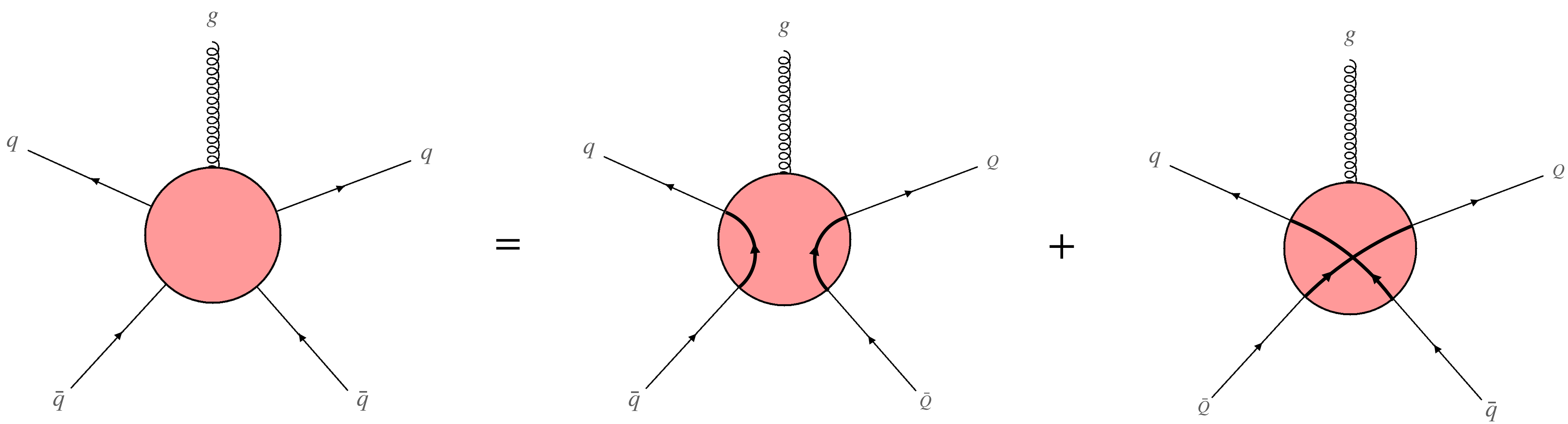
- once and for all
- full information
- infinities are regulated

- more complex structures

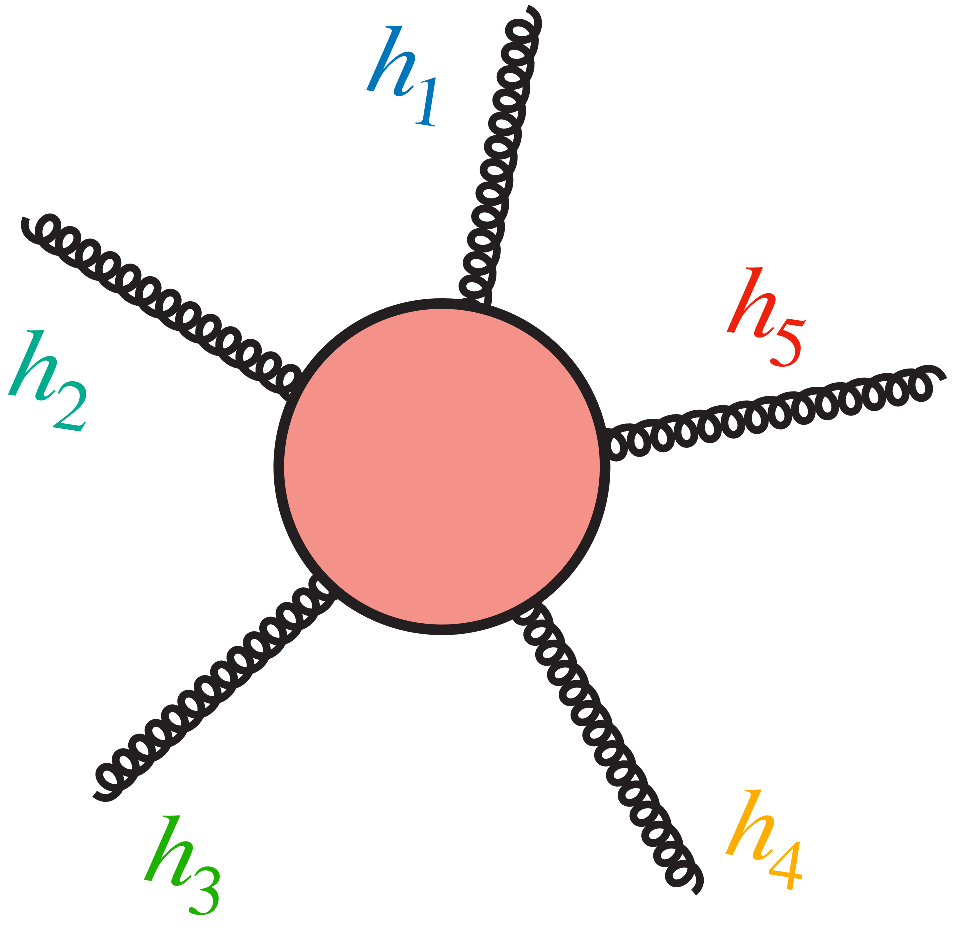
A Colour Game

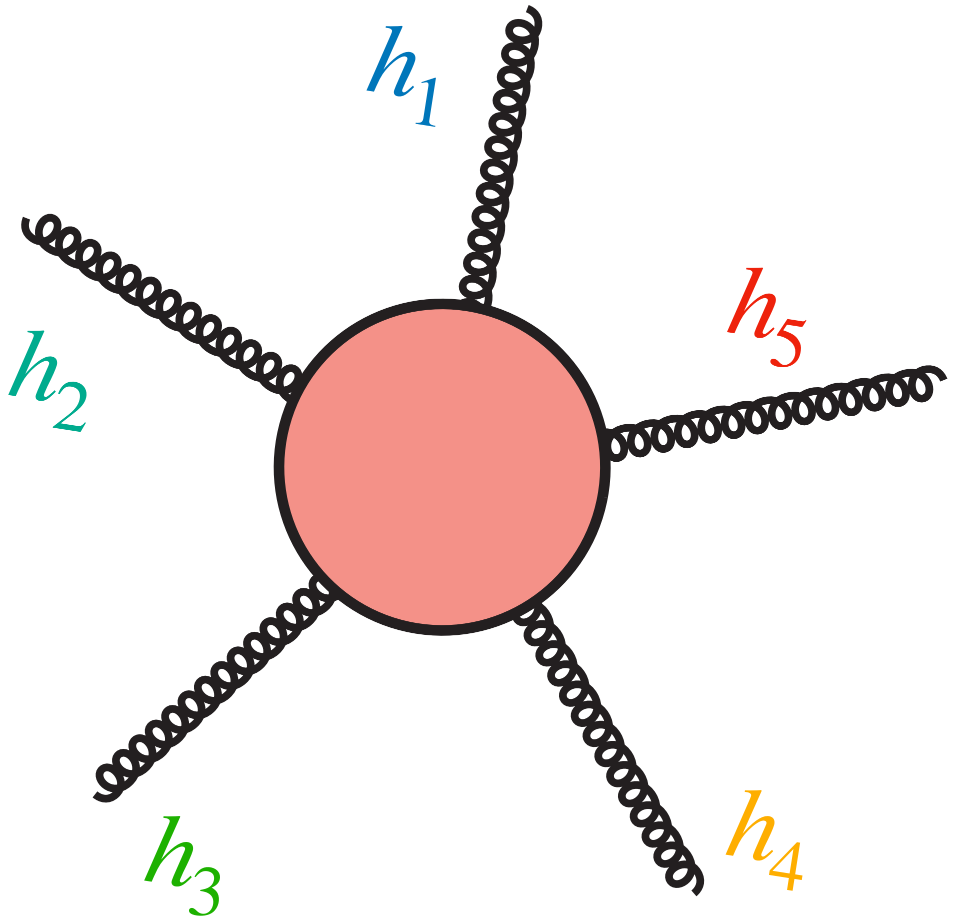






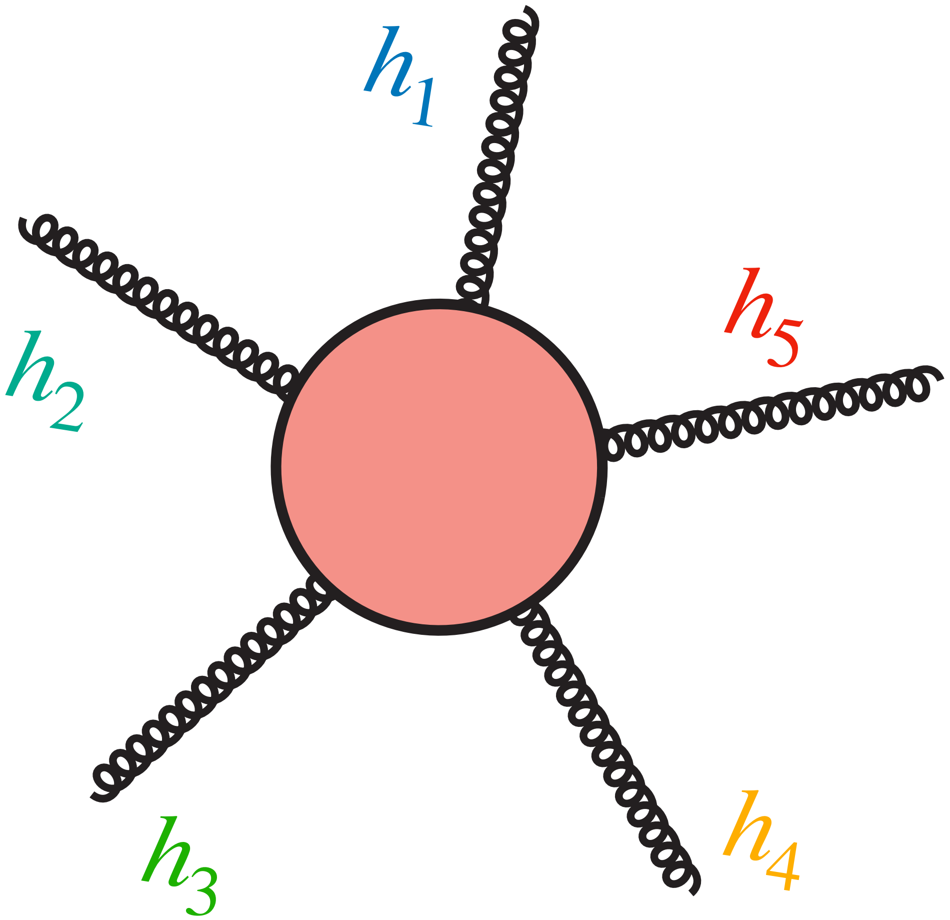
g
q
q
q̄
q̄
=
g
q
Q
q̄
Q̄
+
g
q
Q
Q̄
q̄


h_1
h_2
h_3
h_4
h_5



$$A^{h_1 h_2 \ldots h_5} =$$

$$A^{\mu_1 \mu_2 \ldots \mu_5} \, \epsilon_{\mu_1}^{h_1} \ldots \epsilon_{\mu_5}^{h_5}$$



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Projectors &
Form Factors

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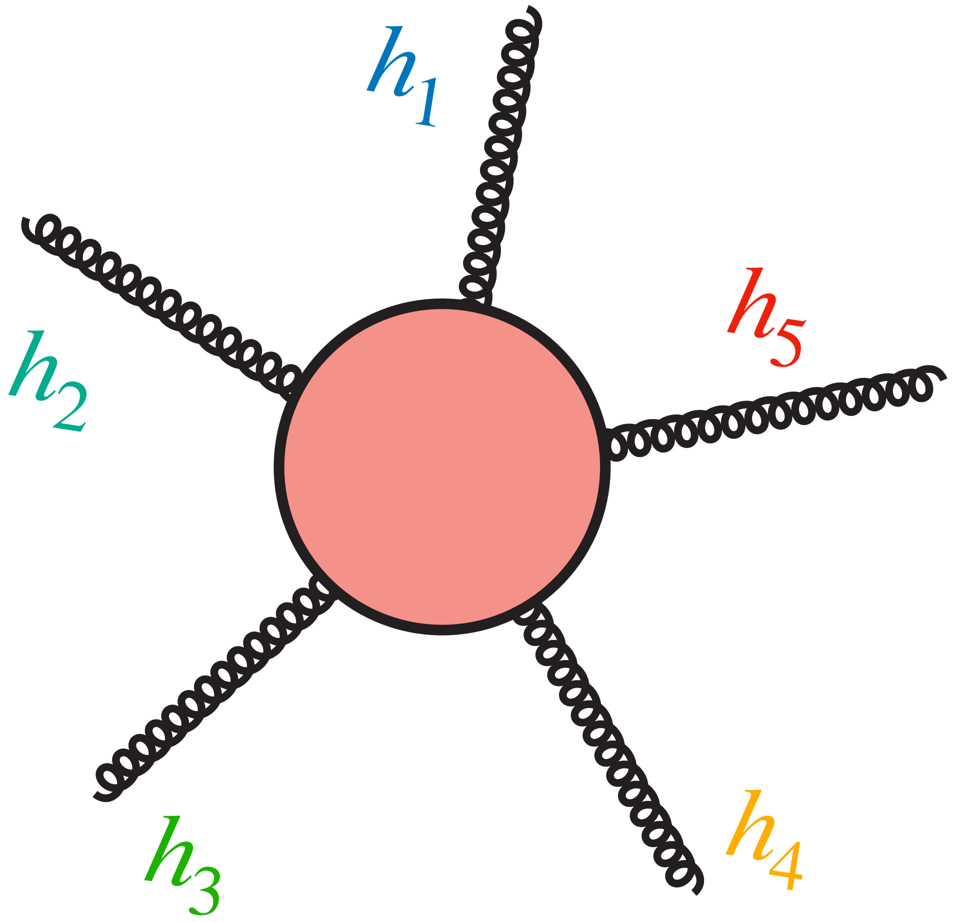
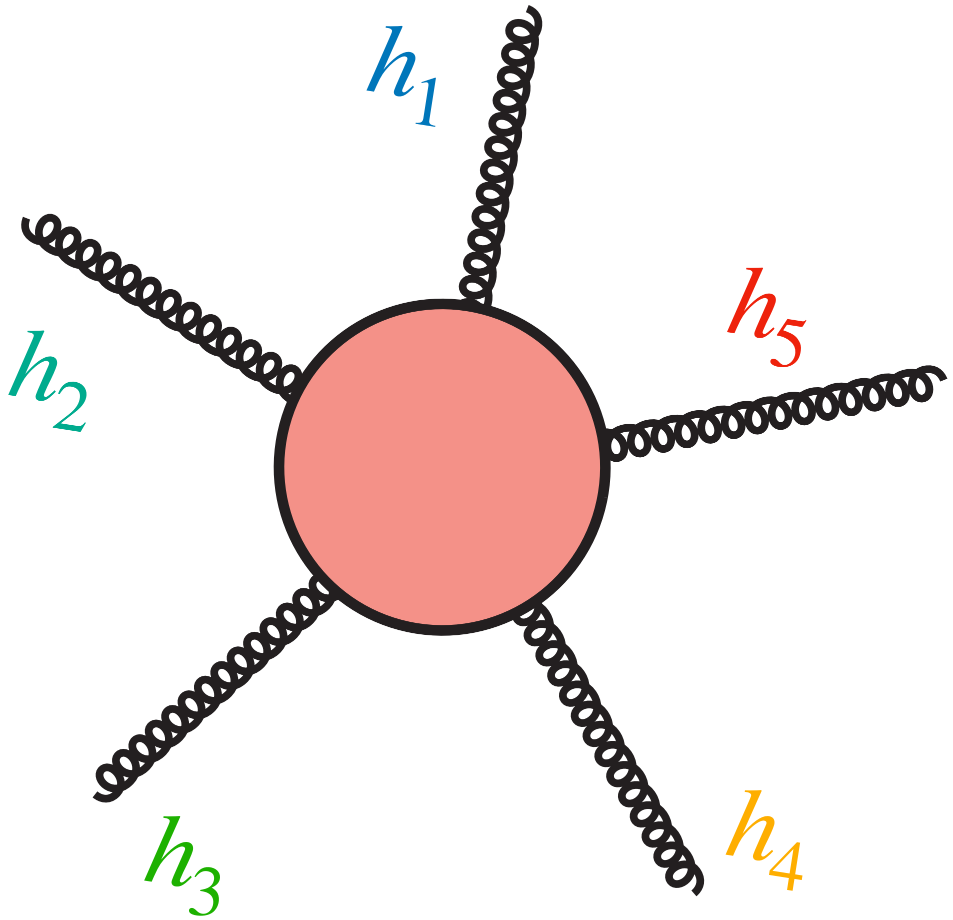


$$A^{h_1 h_2 \ldots h_5} = A^{\mu_1 \mu_2 \ldots \mu_5} \, \epsilon_{\mu_1}^{h_1} \ldots \epsilon_{\mu_5}^{h_5}$$

Projectors &
Form Factors

$$A^{h_1 h_2 \ldots h_5} = \sum_{j=1}^{32} F^j \, T_j^{\mu_1 \ldots \mu_5} \, \epsilon_{\mu_1}^{h_1} \ldots \epsilon_{\mu_5}^{h_5}$$



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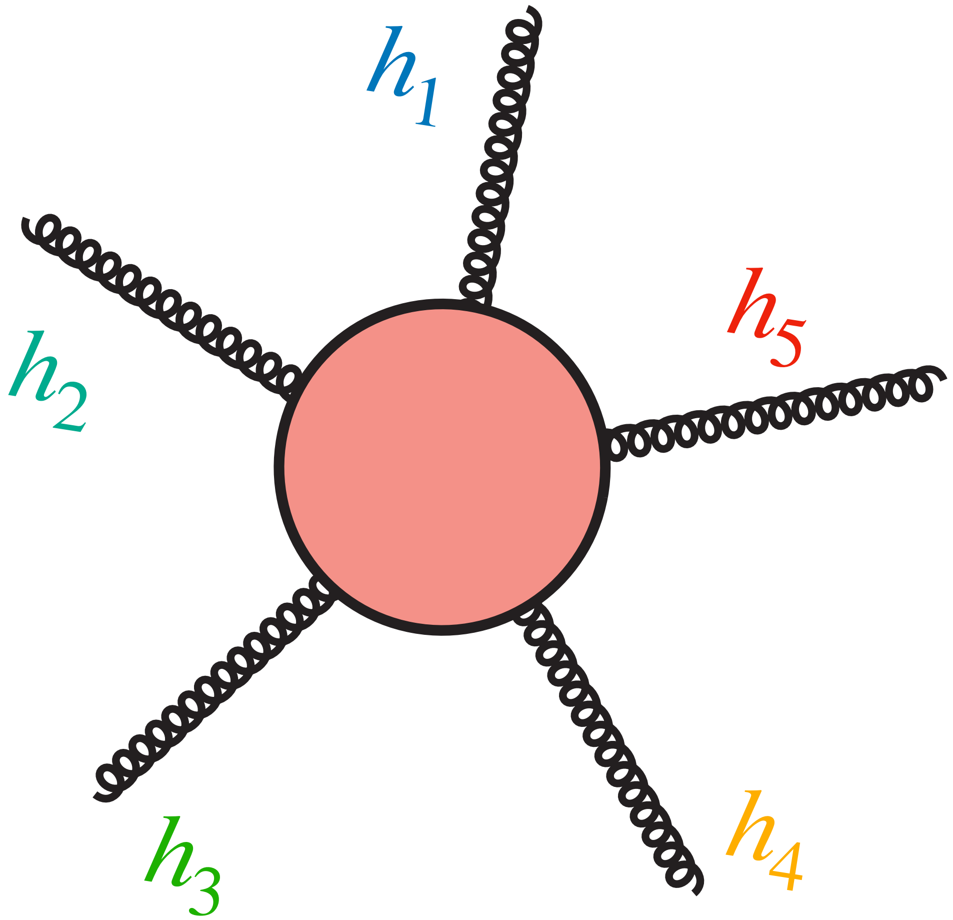

$$A^{h_1 h_2 \ldots h_5} = A^{\mu_1 \mu_2 \ldots \mu_5} \, \epsilon_{\mu_1}^{h_1} \ldots \epsilon_{\mu_5}^{h_5}$$

Form Factors

Projectors &
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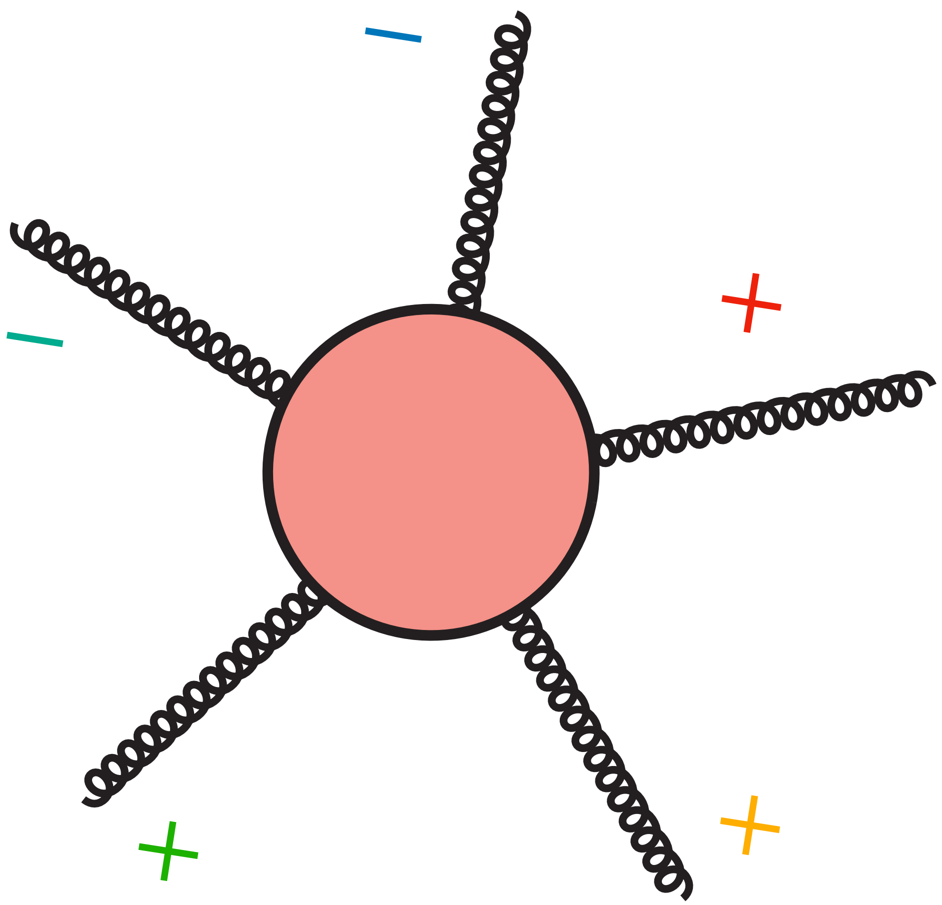
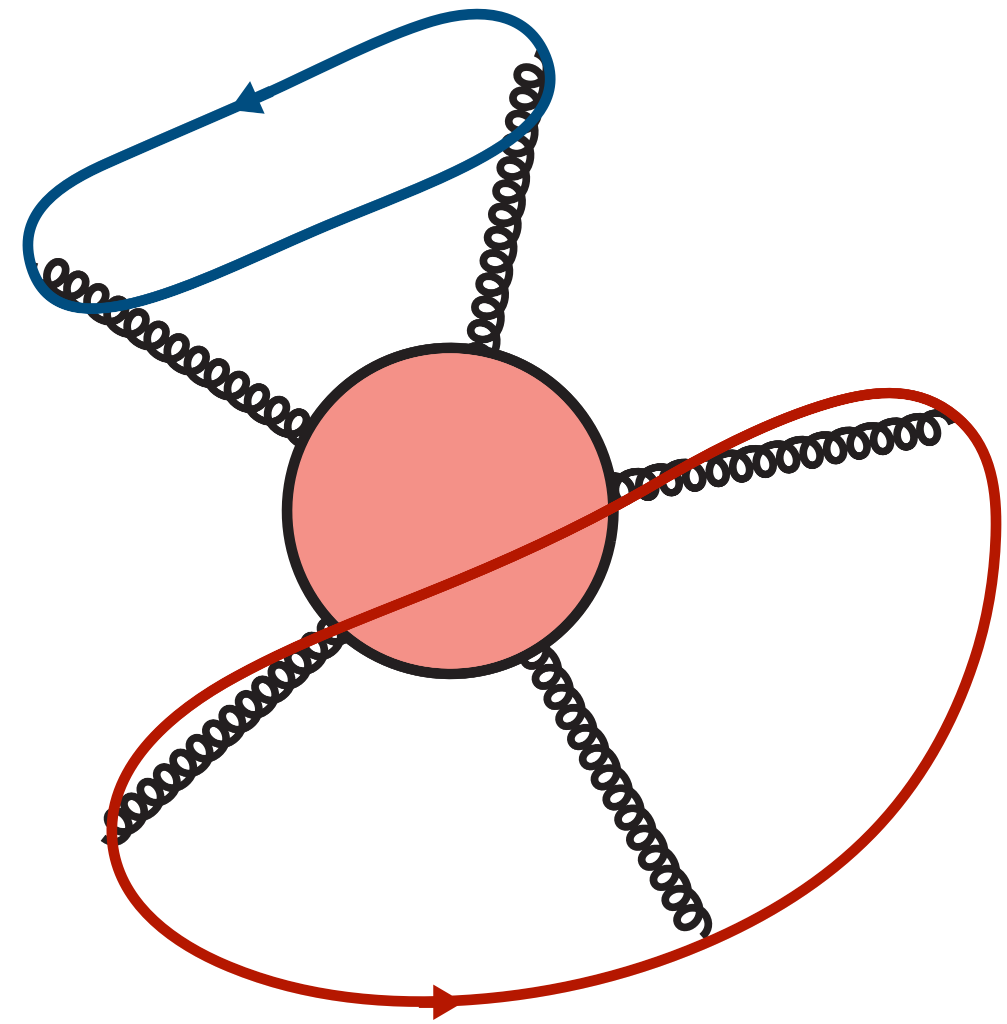
$$A^{h_1 h_2 \ldots h_5} = \sum_{j=1}^{32} F^j T_j^{\mu_1 \ldots \mu_5} \, \epsilon_{\mu_1}^{h_1} \ldots \epsilon_{\mu_5}^{h_5}$$

$$= \sum_{j=1}^{32} F^j \, \Phi_j^{h_1 h_2 \ldots h_5}$$

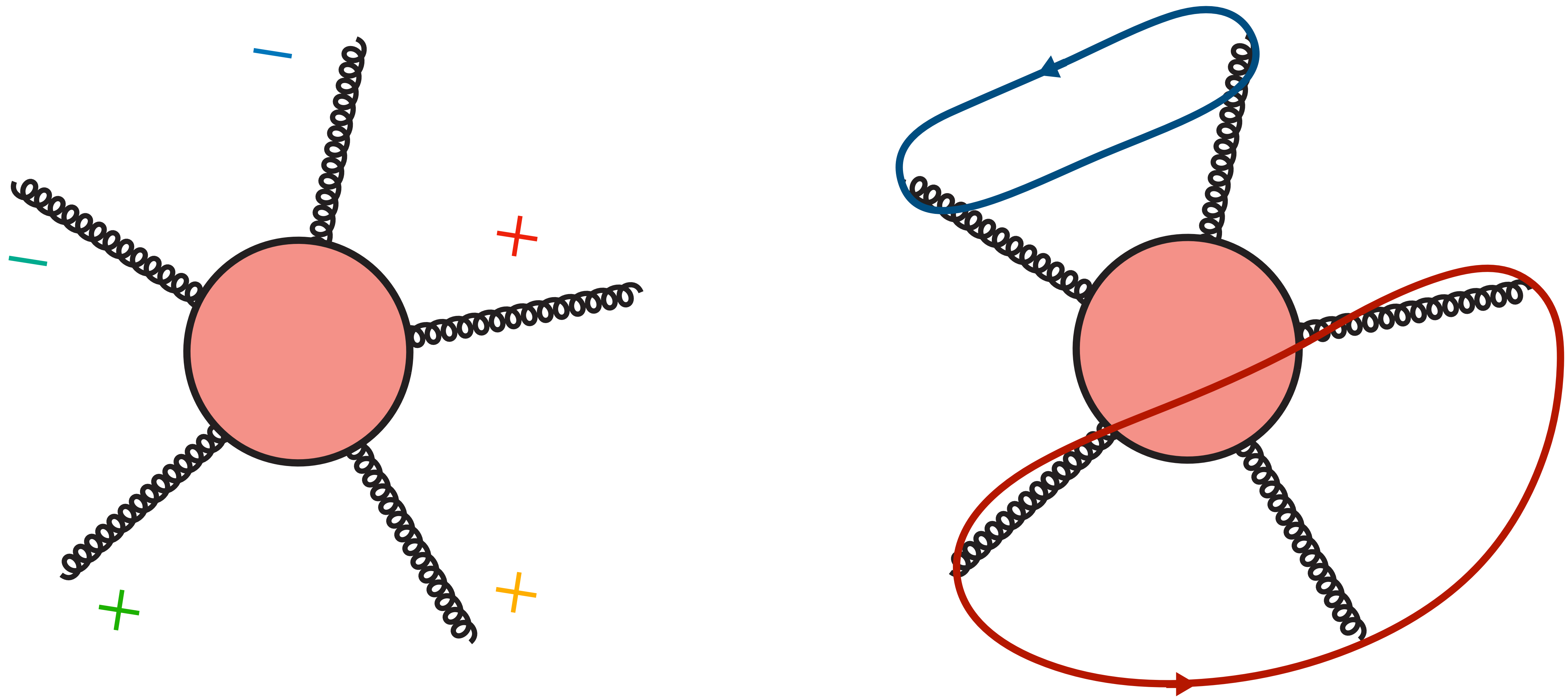
Example 2





$$\epsilon^-_{\mu_1}\epsilon^-_{\mu_2}\ldots\epsilon^+_{\mu_5} = \frac{1}{2^{5/2}}\frac{\langle q_1|\mu_1|1]\,\langle q_2|\mu_1|2]\,[q_3|\mu_1|3\rangle\,[q_4|\mu_1|4\rangle\,[q_5|\mu_1|5\rangle}{\langle q_1p_1\rangle\,\langle q_2p_2\rangle\,[q_3p_3]\,[q_4p_4]\,[q_5p_5]}$$

Example 2



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$$\epsilon^-_{\mu_1}\epsilon^-_{\mu_2}\ldots\epsilon^+_{\mu_5} = \frac{1}{2^{5/2}}\frac{\mathrm{Tr}_+\left\{\gamma_{\mu_1}p_1\gamma_{\mu_2}p_2\right\}\mathrm{Tr}_-\left\{\gamma_{\mu_3}p_3\gamma_{\mu_4}p_4\gamma_{\mu_5}p_5\right\}}{\langle 12\rangle\,\langle 21\rangle\,[34]\,[45]\,[53]}$$

$$A^{\mu_1 \dots \mu_5} \, \epsilon^{h_1}_{\bar{\mu}_1} \dots \epsilon^{h_5}_{\bar{\mu}_5}$$

$$\underbrace{A^{\mu_1\ldots\mu_5}}_{D=4-2\epsilon}\;\underbrace{\epsilon^{h_1}_{\bar{\mu}_1}\ldots\epsilon^{h_5}_{\bar{\mu}_5}}_{D=4}$$

't Hooft-Veltman
scheme

$$A^{\mu_1 \ldots \mu_5} \, \epsilon^{h_1}_{\bar{\mu}_1} \ldots \epsilon^{h_5}_{\bar{\mu}_5}$$

$D = 4 - 2\epsilon$ $\qquad$ $D = 4$

't Hooft-Veltman scheme

p^i_μ

$\bar{g}_{\bar{\mu}\bar{\nu}}$

$\epsilon_{\mu\nu\rho\sigma}$

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$D = 4 - 2\epsilon$ $\quad$ $D = 4$

't Hooft-Veltman scheme

p^i_μ ✓

$$\bar{g}_{\bar{\mu}\bar{\nu}} = g_{\mu\nu} - \frac{1}{G(1234)} G \begin{pmatrix} \mu & p_1 & p_2 & p_3 & p_4 \\ \nu & p_1 & p_2 & p_3 & p_4 \end{pmatrix}$$ ✓

$\epsilon_{\mu\nu\rho\sigma}$

$$A^{\mu_1\ldots\mu_5}\,\epsilon^{h_1}_{\bar{\mu}_1}\ldots\epsilon^{h_5}_{\bar{\mu}_5}$$

$\underbrace{\qquad\qquad}_{D=4-2\epsilon}$ $\underbrace{\qquad\qquad}_{D=4}$

't Hooft-Veltman scheme

p^i_μ ✓

$$\bar{g}_{\bar{\mu}\bar{\nu}} = g_{\mu\nu} - \frac{1}{G(1234)} G\begin{pmatrix} \mu & p_1 & p_2 & p_3 & p_4 \\ \nu & p_1 & p_2 & p_3 & p_4 \end{pmatrix}$$ ✓

$$\epsilon_{\mu\nu\rho\sigma} = \frac{1}{\epsilon_{1234}}\,\epsilon_{\mu\nu\rho\sigma}\epsilon_{1234}$$

$$A^{\mu_1 \ldots \mu_5} \, \epsilon^{h_1}_{\bar{\mu}_1} \ldots \epsilon^{h_5}_{\bar{\mu}_5}$$

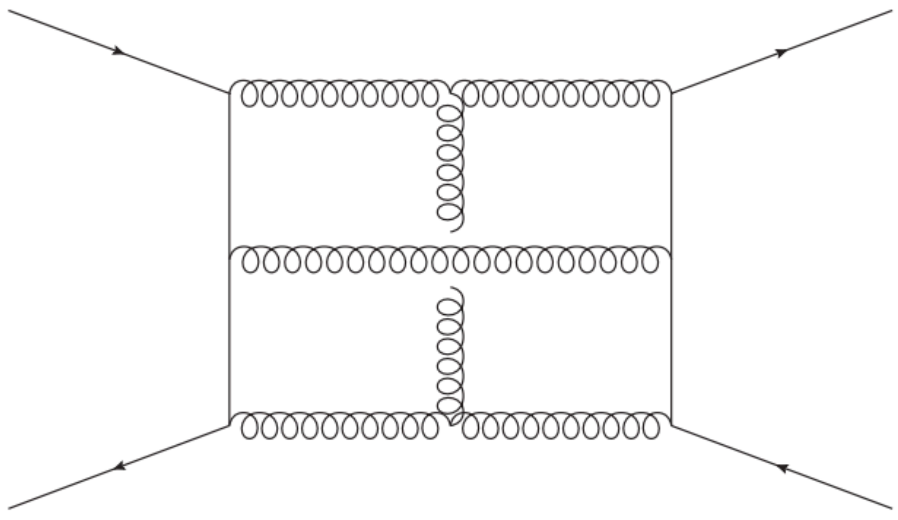
$D = 4 - 2\epsilon$ $\quad$ $D = 4$

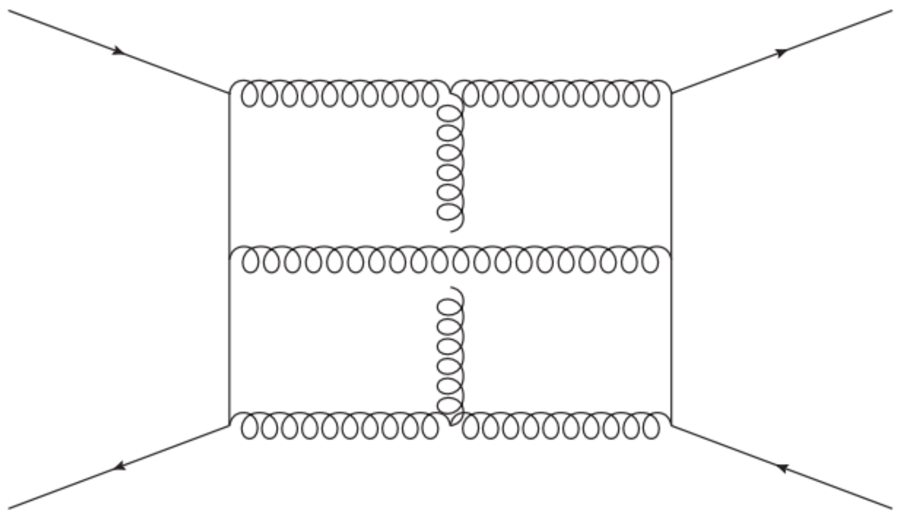
't Hooft-Veltman scheme

$$p^i_\mu \quad ✓$$

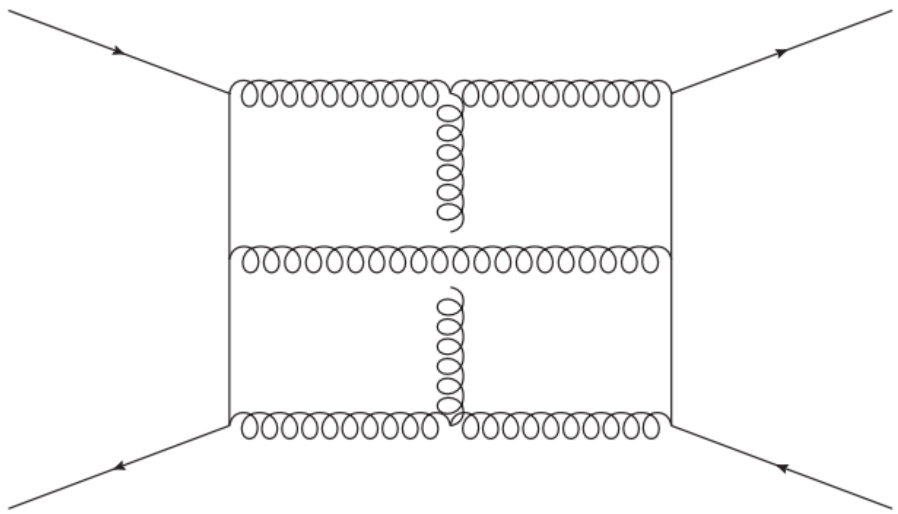
$$\bar{g}_{\bar{\mu}\bar{\nu}} = g_{\mu\nu} - \frac{1}{G(1234)} G\begin{pmatrix} \mu & p_1 & p_2 & p_3 & p_4 \\ \nu & p_1 & p_2 & p_3 & p_4 \end{pmatrix} \quad ✓$$

$$\epsilon_{\mu\nu\rho\sigma} = \frac{1}{\epsilon_{1234}} \epsilon_{\mu\nu\rho\sigma} \epsilon_{1234} = \frac{1}{\epsilon_{1234}} \left(\ldots + p_i^\mu p_j^\nu p_k^\rho p_h^\sigma + \ldots \right) \quad ✓$$





$$\mathcal{C}_1 = \delta_{i_1 i_4}\delta_{i_2 i_3} \ , \qquad \mathcal{C}_2 = \delta_{i_1 i_2}\delta_{i_3 i_4}$$



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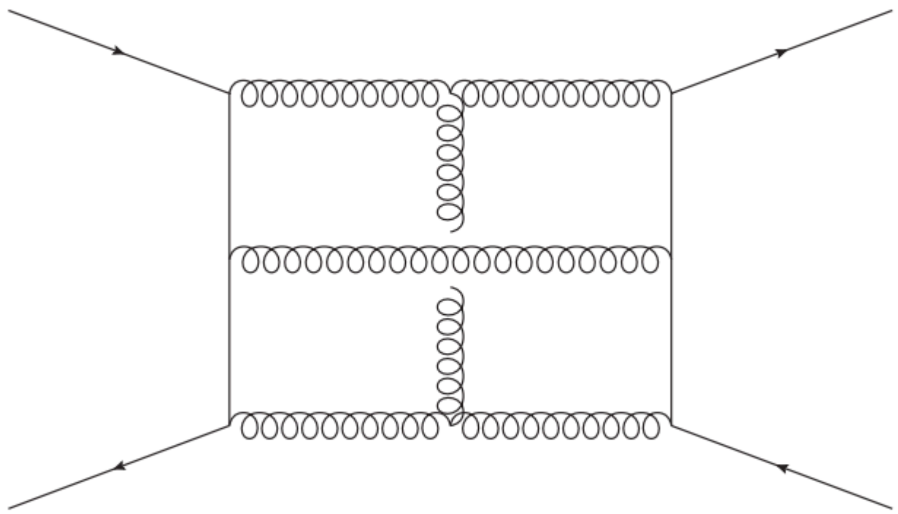
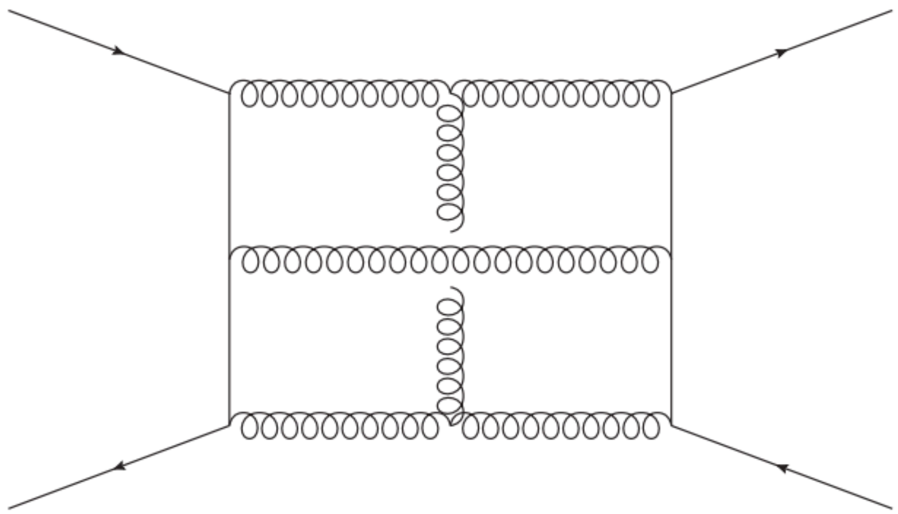
$$(\mathbf{T}_1^a \; \mathbf{T}_2^a \; \mathcal{C}_1)_{i_1 i_2 i_3 i_4} = -T^a_{j_1 i_1} T^a_{i_2 j_2} \; (\delta_{j_1 i_4}\delta_{j_2 i_3}) = -\; \frac{1}{2}\left(\delta_{i_1 i_2}\delta_{i_3 i_4} - \frac{1}{N_c}\delta_{i_1 i_4}\delta_{i_2 i_3}\right)$$



$$\mathcal{C}_1 = \delta_{i_1 i_4}\delta_{i_2 i_3} \ , \qquad \mathcal{C}_2 = \delta_{i_1 i_2}\delta_{i_3 i_4}$$

$$(\mathbf{T}_1^a \ \mathbf{T}_2^a \ \mathcal{C}_1)_{i_1 i_2 i_3 i_4} = -T^a_{j_1 i_1} T^a_{i_2 j_2} \ (\delta_{j_1 i_4}\delta_{j_2 i_3}) = - \ \frac{1}{2}\left(\delta_{i_1 i_2}\delta_{i_3 i_4} - \frac{1}{N_c}\delta_{i_1 i_4}\delta_{i_2 i_3}\right)$$

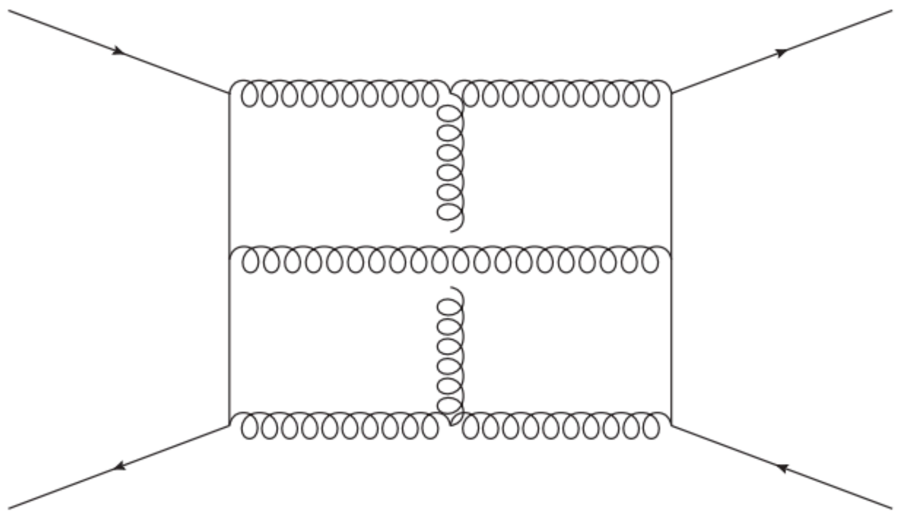
$$(\mathbf{T}_1^a \ \mathbf{T}_2^a \ \mathcal{C}_2)_{i_1 i_2 i_3 i_4} = -T^a_{j_1 i_1} T^a_{i_2 j_2} \ (\delta_{j_1 j_2}\delta_{i_3 i_4}) = -C_F \ \delta_{i_1 i_2}\delta_{i_3 i_4}$$



$$\mathcal{C}_1 = \delta_{i_1 i_4}\delta_{i_2 i_3}\ , \qquad \mathcal{C}_2 = \delta_{i_1 i_2}\delta_{i_3 i_4}$$

$$(\mathbf{T}_1^a\ \mathbf{T}_2^a\ \mathcal{C}_1)_{i_1 i_2 i_3 i_4} = -T^a_{j_1 i_1} T^a_{i_2 j_2}\ (\delta_{j_1 i_4}\delta_{j_2 i_3}) = -\ \frac{1}{2}\left(\delta_{i_1 i_2}\delta_{i_3 i_4} - \frac{1}{N_c}\delta_{i_1 i_4}\delta_{i_2 i_3}\right)$$

$$(\mathbf{T}_1^a\ \mathbf{T}_2^a\ \mathcal{C}_2)_{i_1 i_2 i_3 i_4} = -T^a_{j_1 i_1} T^a_{i_2 j_2}\ (\delta_{j_1 j_2}\delta_{i_3 i_4}) = -C_F\ \delta_{i_1 i_2}\delta_{i_3 i_4}$$

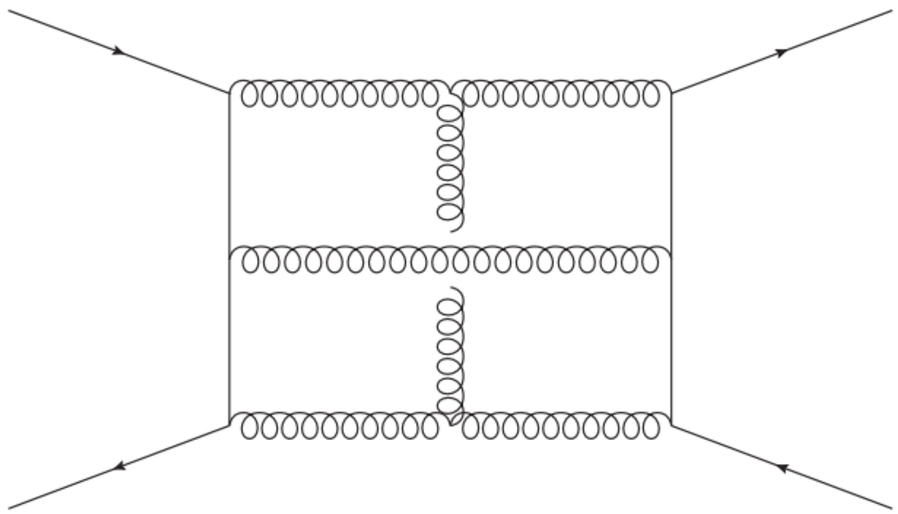


$$\mathcal{C}_1 = \delta_{i_1 i_4}\delta_{i_2 i_3}\ , \qquad \mathcal{C}_2 = \delta_{i_1 i_2}\delta_{i_3 i_4}$$

$$(\mathbf{T}_1^a\ \mathbf{T}_2^a\ \mathcal{C}_1)_{i_1 i_2 i_3 i_4} = -T^a_{j_1 i_1} T^a_{i_2 j_2}\ (\delta_{j_1 i_4}\delta_{j_2 i_3}) = -\ \frac{1}{2}\left(\delta_{i_1 i_2}\delta_{i_3 i_4} - \frac{1}{N_c}\delta_{i_1 i_4}\delta_{i_2 i_3}\right)$$

$$(\mathbf{T}_1^a\ \mathbf{T}_2^a\ \mathcal{C}_2)_{i_1 i_2 i_3 i_4} = -T^a_{j_1 i_1} T^a_{i_2 j_2}\ (\delta_{j_1 j_2}\delta_{i_3 i_4}) = -C_F\ \delta_{i_1 i_2}\delta_{i_3 i_4}$$

$$v_1\ \delta_{i_1 i_4}\delta_{i_2 i_3} + v_2\ \delta_{i_1 i_2}\delta_{i_3 i_4} \to \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$



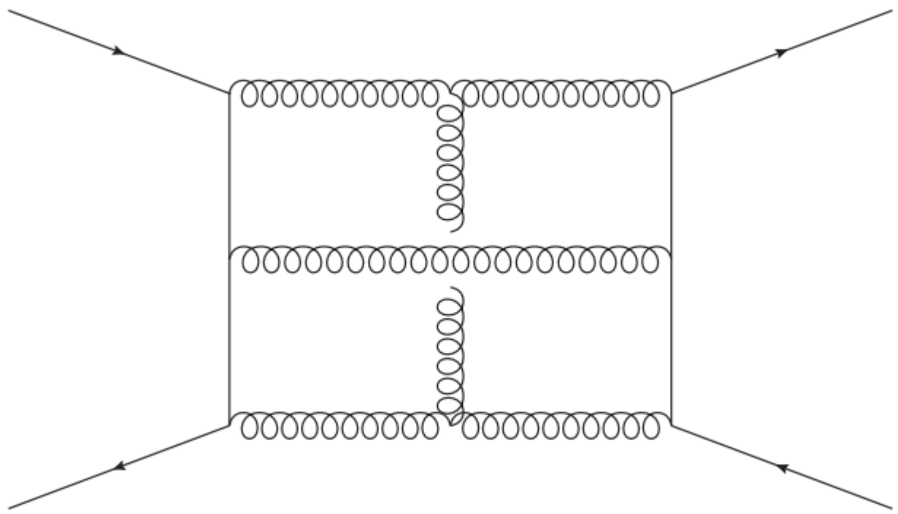
$$\mathcal{C}_1 = \delta_{i_1 i_4}\delta_{i_2 i_3}\ , \qquad \mathcal{C}_2 = \delta_{i_1 i_2}\delta_{i_3 i_4}$$

$$(\mathbf{T}_1^a\ \mathbf{T}_2^a\ \mathcal{C}_1)_{i_1 i_2 i_3 i_4} = -T^a_{j_1 i_1} T^a_{i_2 j_2}\ (\delta_{j_1 i_4}\delta_{j_2 i_3}) = -\ \frac{1}{2}\left(\delta_{i_1 i_2}\delta_{i_3 i_4} - \frac{1}{N_c}\delta_{i_1 i_4}\delta_{i_2 i_3}\right)$$

$$(\mathbf{T}_1^a\ \mathbf{T}_2^a\ \mathcal{C}_2)_{i_1 i_2 i_3 i_4} = -T^a_{j_1 i_1} T^a_{i_2 j_2}\ (\delta_{j_1 j_2}\delta_{i_3 i_4}) = -C_F\ \delta_{i_1 i_2}\delta_{i_3 i_4}$$

$$v_1\ \delta_{i_1 i_4}\delta_{i_2 i_3} + v_2\ \delta_{i_1 i_2}\delta_{i_3 i_4} \rightarrow \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\mathbf{T}_1^a\ \mathbf{T}_2^a = \begin{pmatrix} \frac{1}{2N_c} & 0 \\ -\frac{1}{2} & -C_F \end{pmatrix}$$



$$\mathcal{C}_1 = \delta_{i_1 i_4}\delta_{i_2 i_3}\ , \qquad \mathcal{C}_2 = \delta_{i_1 i_2}\delta_{i_3 i_4}$$

$$(\mathbf{T}_1^a\ \mathbf{T}_2^a\ \mathcal{C}_1)_{i_1i_2i_3i_4} = -T^a_{j_1i_1}T^a_{i_2j_2}\ (\delta_{j_1i_4}\delta_{j_2i_3}) = -\ \frac{1}{2}\left(\delta_{i_1i_2}\delta_{i_3i_4} - \frac{1}{N_c}\delta_{i_1i_4}\delta_{i_2i_3}\right)$$

$$(\mathbf{T}_1^a\ \mathbf{T}_2^a\ \mathcal{C}_2)_{i_1i_2i_3i_4} = -T^a_{j_1i_1}T^a_{i_2j_2}\ (\delta_{j_1j_2}\delta_{i_3i_4}) = -C_F\ \delta_{i_1i_2}\delta_{i_3i_4}$$

$$v_1\ \delta_{i_1i_4}\delta_{i_2i_3} + v_2\ \delta_{i_1i_2}\delta_{i_3i_4} \to \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$\mathbf{T}_1^a\ \mathbf{T}_2^a = \begin{pmatrix} \frac{1}{2N_c} & 0 \\ -\frac{1}{2} & -C_F \end{pmatrix}$$

$$\mathbf{T}_1^a\,\mathbf{T}_2^a\,\mathbf{A} =$$

$$\mathbf{T}_1^a\,\mathbf{T}_2^a\,\mathbf{A} =$$

$$\begin{pmatrix}
-\frac{N_c}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} \\
0 & -\frac{N_c}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & -\frac{N_c}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & -\frac{N_c}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{N_c}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{N_c}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{1}{2} \\
-\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & -N_c & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{N_c}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & -\frac{N_c}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & -\frac{N_c}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \\ A_8 \\ A_9 \\ A_{10} \\ A_{11} \\ A_{12} \\ A_{13} \\ A_{14} \\ A_{15} \\ A_{16} \\ A_{17} \\ A_{18} \\ A_{19} \\ A_{20} \\ A_{21} \\ A_{22}
\end{pmatrix}$$

Ideal

ring $\mathscr{R}$

ideal $\mathscr{I}$

$$\mathscr{I} \subseteq \mathscr{R}$$

and

$$i \in \mathscr{I}, r \in \mathscr{R}$$

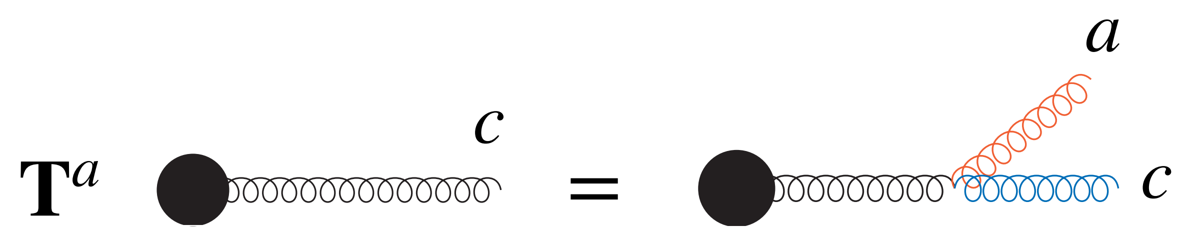
$$r \times i \in \mathscr{I}$$

Example

$$\mathscr{R} = \mathbb{R}[x]$$

$$\mathscr{I} = \langle x^2 \rangle$$

$$i = 3x^2 - 5x^3 = (3 - 5x)x^2$$



$$\mathbf{T}^a \, X^c = -\, i f^a{}_{cc'} \, X^{c'}$$

$$\boldsymbol{\mathcal{I}}_1 = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_i \left(\frac{1}{\epsilon^2} - \frac{1}{2\epsilon} \frac{\gamma_i^{(0)}}{C_i} \right) \sum_{j \neq i} \frac{\mathbf{T}_i^a \mathbf{T}_j^a}{2} \left(-\frac{\mu^2}{s_{ij}} \right)^{\epsilon}$$

$$\boldsymbol{\mathcal{I}}_2 = \frac{-e^{\epsilon\gamma_E}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(\frac{\gamma_K^{(1)}}{8} + \frac{\beta_0}{2\epsilon} \right) \boldsymbol{\mathcal{I}}_1(2\epsilon) - \frac{1}{2} \boldsymbol{\mathcal{I}}_1(\epsilon) \left(\boldsymbol{\mathcal{I}}_1(\epsilon) + \frac{\beta_0}{\epsilon} \right) + \frac{e^{\epsilon\gamma_E}}{4\epsilon\Gamma(1-\epsilon)} \sum_i H_2^i$$

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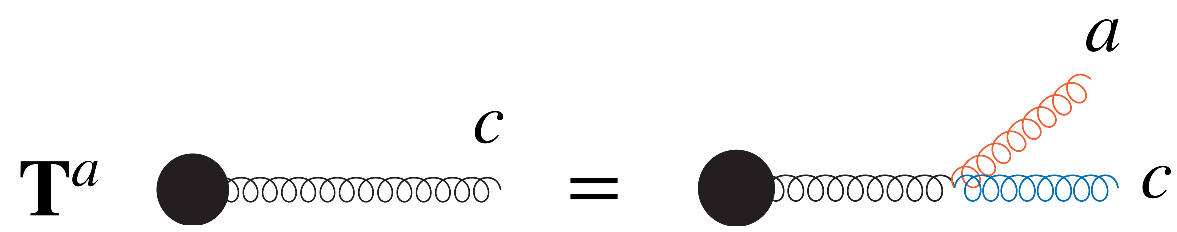

$$\mathbf{T}^a X^c = -i f^a{}_{cc'} X^{c'}$$

$$\boldsymbol{\mathcal{I}}_1 = \frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_i \left(\frac{1}{\epsilon^2} - \frac{1}{2\epsilon} \frac{\gamma_i^{(0)}}{C_i} \right) \sum_{j \neq i} \frac{\mathbf{T}_i^a \mathbf{T}_j^a}{2} \left(-\frac{\mu^2}{s_{ij}} \right)^{\epsilon}$$

$$\boldsymbol{\mathcal{I}}_2 = \frac{-e^{\epsilon\gamma_E}\Gamma(1-2\epsilon)}{\Gamma(1-\epsilon)} \left(\frac{\gamma_K^{(1)}}{8} + \frac{\beta_0}{2\epsilon} \right) \boldsymbol{\mathcal{I}}_1(2\epsilon) - \frac{1}{2}\boldsymbol{\mathcal{I}}_1(\epsilon) \left(\boldsymbol{\mathcal{I}}_1(\epsilon) + \frac{\beta_0}{\epsilon} \right) + \frac{e^{\epsilon\gamma_E}}{4\epsilon\Gamma(1-\epsilon)} \sum_i H_2^i$$

leading-colour → diagonal in colour-space

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Rearranging the Amplitudes

$$\mathcal{H} = \sum_t \left(\sum_i r_i\right) T_t$$

"Master" rational functions $$r_i = \sum_r c_r R_r$$

O(10) Reduction

$$\mathcal{H} = \sum_r \left(\sum_t a_t T_t\right) R_r$$