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IN CONJUNCTION WITH ETH ZÜRICH

Applications of General Relativity in Astrophysics and Cosmology

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Contents

Preface

This course is a continuation of the General Relativity course taught by Prof. Philippe Jetzer in fall 2013. The script covering this course can be found on the webpage http://www.physik.uzh.ch/lectures/agr/. When referring to equations in the General Relativity script, we use the prefix GR I , as for example in Eq. (GR I 1.1). These lecture notes are far from being a complete treatment of the subject but should enable the student or interested reader to access specialized literature on the subject. For further reading we included a list of useful textbooks at the end. If you find any mistakes, please report them to balmelli@physik.uzh.ch.

Zürich, February 2014

1 Time Delay of Radar Echoes

In 1964 Shapiro proposed a new test of GR consisting of a measurement of the time delay of radar signals transmitted from Earth through a region near the Sun to another planet or spacecraft and reflected back to Earth. Since the radar signal is affected by the gravitational field of the Sun, it will return to Earth with a time delay.

The radar signal is emitted from Earth and then sent back by a reflector as sketched in Fig. 1. In calculations we set $\theta = \frac{\pi}{2}$. The trajectory of the signal is curved. This is not shown in the figure because the curvature is, of course, only a tiny effect and the light ray seems almost straight. Let's compute the time that the signal needs to get from Earth to the reflector. To this end we use Eq. (GR I 23.15) for the orbit in a static isotropic gravitational field with $\varepsilon = 0$ (since $m = 0$):

$$A \left(\frac{dr}{d\lambda}\right)^2 + \frac{l^2}{r^2} - \frac{F^2}{B} = 0. \tag{1.1}$$

If we divide this by l^2 and use $x^0 = ct$, we find

$$\frac{dr}{d\lambda} = \frac{1}{c}\frac{dr}{dt}\frac{dx^0}{d\lambda} \overset{\text{(GR I 23.12)}}{=} \frac{1}{c}\frac{dr}{dt}\frac{F}{B}. \tag{1.2}$$

This yields

$$\frac{AF^2}{c^2B^2l^2}\left(\frac{dr}{dt}\right)^2 + \frac{1}{r^2} - \frac{F^2}{Bl^2} = 0. \tag{1.3}$$

For the minimal distance r_0 (from the Sun) it holds that

$$\frac{dr}{dt} = 0 \quad \Rightarrow \quad \frac{F^2}{l^2} = \frac{B(r_0)}{r_0^2}. \tag{1.4}$$

We insert this into Eq. (1.3) to obtain

$$\frac{A}{c^2B}\left(\frac{dr}{dt}\right)^2 + \frac{r_0^2}{r^2}\frac{B(r)}{B(r_0)} - 1 = 0. \tag{1.5}$$

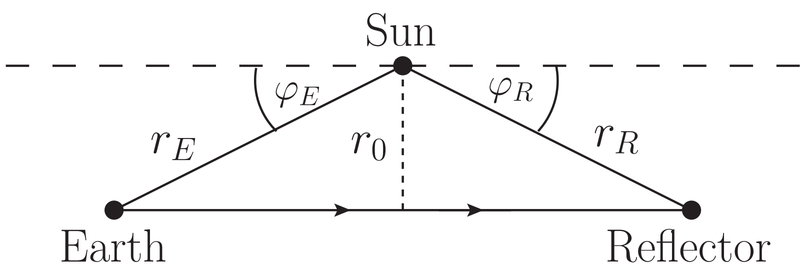

Figure 1: Sketch of the system as described in the text. The angles φ_E and φ_R and the distances r_E, r_R describe the position of the Earth and the reflector relative to the Sun. The whole setup is chosen to lie in the $\theta = \frac{\pi}{2}$-plane.

This differential equation is solved by the following integral:

$$t(r, r_0) = \frac{1}{c}\int_{r_0}^{r} dr' \sqrt{\frac{A}{B}} \left[1 - \frac{r_0^2}{r'^2}\frac{B(r')}{B(r_0)}\right]^{-1/2} , \tag{1.6}$$

where $t(r, r_0)$ is the time that the radar signal needs to travel from r_0 to r. Note that this is the time which would be shown by a clock at rest at infinity (as space is asymptotically Minkowskian at infinity). This actually forces us to introduce a correction since our clock rests at Earth, not at infinity. However, the correction which is needed to compensate this effect is much smaller than the time delay and can thus be neglected.

Using the Robertson expansion from Eq. (GR I 22.3),

$$A(r) = 1 + \gamma\frac{2a}{r} + ..., \qquad B(r) = 1 - \frac{2a}{r} + ..., \tag{1.7}$$

we get

$$\begin{aligned} 1 - \frac{r_0^2}{r^2}\frac{B(r)}{B(r_0)} &= 1 - \frac{r_0^2}{r^2}\left[1 + 2a\left(\frac{1}{r_0} - \frac{1}{r}\right)\right] \\ &= \left[1 - \frac{r_0^2}{r^2}\right]\left[1 - \frac{2ar_0}{r(r+r_0)}\right]. \end{aligned} \tag{1.8}$$

Inserting Eqs. (1.7), (1.8) into (1.6) and expanding, we get

$$\begin{aligned} t(r, r_0) &\simeq \frac{1}{c}\int_{r_0}^{r} dr' \left[1 - \frac{r_0^2}{r'^2}\right]^{-1/2} \left(1 + \frac{ar_0}{r'(r'+r_0)} + (1+\gamma)\frac{a}{r'}\right) \\ &= \frac{\sqrt{r^2 - r_0^2}}{c} + \frac{a}{c}\sqrt{\frac{r - r_0}{r + r_0}} + (1+\gamma)\frac{a}{c}\log\left(\frac{r + \sqrt{r^2 - r_0^2}}{r_0}\right). \end{aligned} \tag{1.9}$$

The first term $\frac{\sqrt{r^2 - r_0^2}}{c}$ corresponds to the travelling time assuming a straight trajectory in Euclidean space as can easily be seen from the figure and Pythagoras' theorem. The other terms account for the general relativistic time delay due to the gravitational field of the Sun. For the system drawn in the figure, the total delay is

$$\delta t = 2\left[t(r_{\mathrm{E}}, r_0) + t(r_{\mathrm{R}}, r_0) - \frac{\sqrt{r_{\mathrm{E}}^2 - r_0^2}}{c} - \frac{\sqrt{r_{\mathrm{R}}^2 - r_0^2}}{c}\right] , \tag{1.10}$$

where the factor of 2 accounts for the fact that the signal travels from Earth to the reflector and back again.

Significant delays occur if the radar signal passes nearby the Sun, i.e. if r_0 is of the order of some Sun radii. In this case we have $r_{\mathrm{E}}, r_{\mathrm{R}} \gg R_\odot$ and also $r_{\mathrm{E}}, r_{\mathrm{R}} \gg r_0$. We thus can perform the following approximations in Eq. (1.10):[1]

$$\delta t \simeq \frac{4a}{c}\left[1 + \left(\frac{1+\gamma}{2}\right)\log\frac{4r_{\mathrm{E}}r_{\mathrm{R}}}{r_0^2}\right]. \tag{1.11}$$

[1] I. I. Shapiro, Phys. Rev. Lett. **13**, 789 (1964)

We see that δt is maximal if the signal just grazes the surface of the Sun, i.e. $r_0 = R_\odot$.

In order to see the orders of magnitude, we use $r_{\rm E} \sim r_{\rm R} \sim 10^8$ km, $R_\odot \simeq 7 \cdot 10^5$ km. This yields $\frac{2a}{c} = \frac{2GM_\odot}{c^3} \simeq 10^{-5}$ s and thus

$$\begin{aligned}\delta t_{\rm max} &= \frac{4a}{c}\left[1 + \left(\frac{1+\gamma}{2}\right)\log\frac{4r_{\rm E}r_{\rm R}}{R_\odot^2}\right] \\ &\simeq 2 \cdot 10^{-4}\ {\rm s}.\end{aligned} \tag{1.12}$$

Performing the measurements has been a very difficult task since the distances $r_{\rm E}$ and $r_{\rm R}$ were not known with sufficient precision. Nevertheless, in the seventies these measurements were performed using Venus and Mercury and later again using spacecrafts as reflectors (e.g. the Vikings which landed on Mars or, more recently, the Cassini spacecraft[2]). The results are

$$\begin{aligned}&\text{Vikings:} \quad \gamma = 1.000 \pm 0.001, \\ &\text{Cassini:} \quad \gamma = 1 + (2.1 \pm 2.3) \cdot 10^{-5}.\end{aligned} \tag{1.13}$$

Another experimental verification of the Shapiro delay is the measurement of the PSR J1614-2230 system. It consists of a pulsar which emitts signals in very regular time intervals, and a white dwarf that orbits the pulsar. When the white dwarf is in front of the pulsar and the light of the pulsar arrives at Earth by passing close to the white dwarf, then the signal arrives with a delay. Measuring the Shapiro delay[3], one can infer the mass of the white dwarf to be $0.500 \pm 0.006 M_\odot$. With an orbital period of 8.7 days, this yields a neutron star mass of $1.97 \pm 0.04 M_\odot$. This result is important for the modelling of neutron stars since the largest neutron stars (so far) had masses of about $1.4 M_\odot$.

[2]B. Bertotti et al., Nature **425**, 374 (2003)
[3]P. Demorest et al., Nature **467**, 1081 (2010)

2 Geodetic Precession

Consider a particle with a "classical" angular momentum (for instance the intrinsic angular momentum of a rigid body like a gyroscope). In the local inertial system in which the body is at rest, the spin (i.e. angular momentum) is given by $\boldsymbol{S} = S^i \boldsymbol{e}_i$. To the three-vector S^i we assign a Lorentz vector S^α. Consider now a locally inertial coordinate frame IS' which is momentarily at rest with respect to the rigid body (or particle):

$$S'^\alpha = (0, S'^i). \tag{2.1}$$

We can transform this to some arbitrary inertial system IS by means of a Lorentz transformation. In the rest frame IS', the velocity of the body is

$$u'^\alpha = (c, \mathbf{0}).$$

Therefore we have in IS'

$$u'_\alpha S'^\alpha = 0.$$

Since this quantity is a Lorentz scalar, $u_\alpha S^\alpha = 0$ in any arbitrary IS.

Consider first the case without any forces acting on the particle and no torque acting on its spin. In an arbitrary IS we have

$$\frac{dS^\alpha}{d\tau} = 0. \tag{2.2}$$

We define the Riemann vector

$$S^\mu \equiv \frac{\partial x^\mu}{\partial \xi^\alpha} S^\alpha \ , \tag{2.3}$$

which describes the transition from the coordinate system IS with coordinates (ξ^α) to a general system (x^μ). According to the covariance principle, the generalization of Eq. (2.2) reads

$$\frac{DS^\mu}{d\tau} = 0 \qquad \Leftrightarrow \qquad \frac{dS^\mu}{d\tau} = -\Gamma^\mu{}_{\nu\lambda} u^\nu S^\lambda. \tag{2.4}$$

This equation describes the spin precession in a gravitational field. The condition $u_\alpha S^\alpha = 0$ reads $u_\mu S^\mu = 0$ in the general coordinate system. Note that $S_\mu S^\mu$ =const., which implies that (2.4) describes the *rotation* or *precession* of the spin vector. Since we assumed no external forces, Eq. (2.4) contains only gravitational effects. We conclude that Eq. (2.4) describes the precession of the spin of a particle which is freely falling in a gravitational field as, for example, the precession of a rotating satellite (or gyroscope).

If there are other external forces f^μ besides gravity, then one finds instead of Eq. (2.4)

$$\frac{DS^\nu}{d\tau} = -\frac{1}{c^2} \frac{Du^\mu}{d\tau} S_\mu u^\nu \tag{2.5}$$

which is also called **Fermi transport**. It describes the spin precession of an accelerated particle on which a gravitational field acts (c.f. Eq. (GR I 19.2), $Du^\mu/d\tau = f^\mu/m$). (The special case $f^\mu = 0$ is just parallel transport.)

Based on the above considerations, we shall study the following effects (in the gravitational field of the Earth):

1. **Geodetic precession:** the precession of a freely falling gyroscope. In order to simplify the analysis we will assume the gravitational field to be isotropic and static.

2. **Lense-Thirring effect:** the precession of a gyroscope in the gravitational field of the Earth which is due to the rotation of the Earth. This is a smaller effect.

2.1 Geodetic Precession

Gyroscopes are rigid bodies which can also perform rotations described by S^μ besides the movement of its center of mass. To compute the geodetic precession of a gyroscope we use Eq. (2.4). In the local rest frame of the satellite (orbiting the Earth) we have for the spin vector:

$$S'^{\alpha} = (0, \boldsymbol{l}) \ , \tag{2.6}$$

where $\boldsymbol{l}$ describes the angular momentum of the gyroscope. We use the standard form of the static and isotropic metric in spherical coordinates (cf. chapter 22.1 of GR I):

$$x^\mu = (ct, r, \theta, \varphi)$$
$$g = \mathrm{diag}\big(B(r),\ -A(r),\ -r^2,\ -r^2\sin^2\theta\big).$$

Assume that the satellite is on a circular orbit, i.e.

$$r = \text{const.}, \qquad \theta = \frac{\pi}{2}, \qquad \varphi = \omega_0 \tau. \tag{2.7}$$

Therefore the velocity of the satellite reads

$$u^\mu = \frac{dx^\mu}{d\tau} = \big(u^0 = \text{const.},\ 0,\ 0,\ u^3 = \omega_0 = \text{const.}\big)\,. \tag{2.8}$$

If we insert $\theta = \frac{\pi}{2}$ into the Christoffel symbols from Eq. (GR I 22.6), we obtain for the non-zero components

$$\Gamma^1{}_{00} = \frac{B'}{2A}, \quad \Gamma^1{}_{11} = \frac{A'}{2A}, \quad \Gamma^1{}_{22} = -\frac{r}{A}, \quad \Gamma^1{}_{33} = -\frac{r}{A}$$
$$\Gamma^0{}_{01} = \Gamma^0{}_{10} = \frac{B'}{2B}, \quad \Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{r}, \quad \Gamma^3{}_{13} = \Gamma^3{}_{31} = \frac{1}{r}. \tag{2.9}$$

Eq. (2.4) reads then

$$\frac{dS^0}{d\tau} = -\Gamma^0{}_{01} u^0 S^1 \tag{2.10}$$

$$\frac{dS^1}{d\tau} = -\Gamma^1{}_{00} u^0 S^0 - \Gamma^1{}_{33} u^3 S^3 \tag{2.11}$$

$$\frac{dS^2}{d\tau} = 0 \tag{2.12}$$

$$\frac{dS^3}{d\tau} = -\Gamma^3{}_{31} u^3 S^1. \tag{2.13}$$

We can immediately solve the third of these equations:

$$S^2(\tau) = \text{const.} \tag{2.14}$$

this is the component of the spin (or angular momentum) of the gyroscope which is perpendicular to the satellite's orbit (θ-direction) is constant.

Because of r =const., all coefficients of the system of linear differential equations (2.10)-(2.13) are constants. We differentiate Eq. (2.11) with respect to τ and insert (2.10) and (2.13) on the right-hand side. This yields

$$\begin{aligned}\frac{d^2S^1}{d\tau^2} &= \left[\Gamma^1{}_{00}\Gamma^0{}_{01}(u^0)^2 + \Gamma^1{}_{33}\Gamma^3{}_{31}(u^3)^2\right] S^1 \\ &\equiv -\omega^2 S^1. \end{aligned} \tag{2.15}$$

With $(u^3)^2 = \omega_0^2$ and inserting the Christoffel symbols from (2.9), we get

$$\omega^2 = \omega_0^2 \left[-\frac{B'^2}{4AB}\left(\frac{u^0}{u^3}\right)^2 + \frac{1}{A}\right]. \tag{2.16}$$

In order to understand the ratio $\frac{u^0}{u^3}$ better, we look at the equation for the trajectory of the satellite (*geodesic equation*):

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu{}_{\nu\lambda} u^\nu u^\lambda. \tag{2.17}$$

The $\mu = 1$ component of this equation is ($u^1 = 0$)

$$0 = \frac{du^1}{d\tau} = -\Gamma^1{}_{00}(u^0)^2 - \Gamma^1{}_{33}(u^3)^2 \tag{2.18}$$

from which we infer

$$\left(\frac{u^0}{u^3}\right)^2 = \frac{2r}{B'}. \tag{2.19}$$

Inserting this into (2.16), we find

$$\omega = \omega_0 \sqrt{\frac{1}{A}\left(1 - \frac{rB'}{2B}\right)}. \tag{2.20}$$

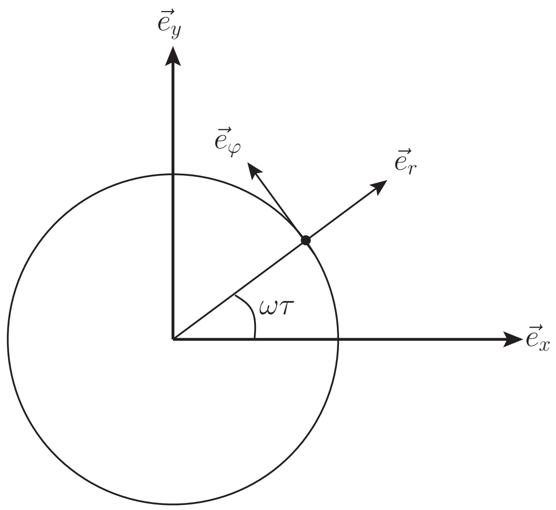

Figure 2: Projection of the motion orbit of precession of the angular momentum vector onto the orbital plane of the satellite. The precession takes place in the (r, φ)-plane since the θ-component of the spin vector (i.e. the component perpendicular to the orbital plane) is constant.

Using the Schwarzschild solution ($B = A^{-1} = 1 - \frac{2a}{r}$), this yields

$$\boxed{\omega = \omega_0 \sqrt{1 - \frac{3a}{r}}}\,, \tag{2.21}$$

or in terms of the Robertson expansion

$$\omega = \omega_0 \sqrt{1 - (1 + 2\gamma)\frac{a}{r}}. \tag{2.22}$$

Since r =const., Eq. (2.15) is of type

$$\ddot{S}^1 + \omega^2 S^1 = 0 \tag{2.23}$$

which describes a harmonic oscillator. With initial conditions $S^1(0) = S$ and $\dot{S}^1(0) = 0$ the solution reads

$$S^1(\tau) = S\cos(\omega\tau), \qquad S^2 = \text{const.}. \tag{2.24}$$

Inserting this into Eq. (2.13), we obtain by integration

$$S^3(\tau) = -\frac{S\omega_0}{r\omega}\sin(\omega\tau). \tag{2.25}$$

We proceed by studying the time dependence of the projection (S^1, S^3) (or (r, φ)-components) of the spin vector onto the orbital plane ($\theta = \frac{\pi}{2}$), c.f. Fig. 2. Consider the constant vector $\boldsymbol{e}_x$ along the orbit of the satellite:

$$\boldsymbol{e}_x = \cos(\underbrace{\omega_0\tau}_{=\varphi})\boldsymbol{e}_r - \sin(\underbrace{\omega_0\tau}_{=\varphi})\boldsymbol{e}_\varphi. \tag{2.26}$$

The orbital period of the satellite is $\tau_0 = \frac{2\pi}{\omega_0}$. After each orbit, the argument φ in (2.26) increases by $\tau_0\omega_0 = 2\pi$ whereas the argument in (2.24) or (2.25) increases by $\tau_0\omega$ which differs slightly from 2π. The phase difference after one orbit is given by (note that in GR $\gamma = 1$)

$$\begin{aligned}\Delta\alpha &= \tau_0(\omega_0 - \omega)\\ &= 2\pi - 2\pi\sqrt{1 - \frac{(1+2\gamma)a}{r}}\\ &\cong \pi\frac{(1+2\gamma)a}{r}.\end{aligned} \tag{2.27}$$

Consider the vector $\boldsymbol{S}$ which is the projection of S^μ onto the orbital plane:

$$\boldsymbol{S} = S_r\boldsymbol{e}_r + S_\varphi\boldsymbol{e}_\varphi. \tag{2.28}$$

The components of this vector are given by

$$(S_r)^2 = -g_{11}S^1S^1 \qquad \text{and} \qquad (S_\varphi)^2 = -g_{33}S^3S^3 \tag{2.29}$$

with

$$-g_{11} = A(r) \qquad \text{and} \qquad -g_{33} = r^2\sin^2\theta\big|_{\theta=\frac{\pi}{2}} = r^2. \tag{2.30}$$

Therefore $S_r \propto \cos(\omega\tau)$ and $S_\varphi \propto \sin(\omega\tau)$. For $\tau = 0$, $\boldsymbol{S}$ is thus parallel to $\boldsymbol{e}_x$. However, after an orbit, $\omega\tau$ differs slightly from 2π as we have calculated in Eq. (2.27). The geodetic precession after one orbit is given by

$$\boxed{\Delta\alpha = \frac{3\pi a}{r}\frac{(1+2\gamma)}{3}}. \tag{2.31}$$

Consider the concrete example of a satellite in a circular orbit around the Earth. We have

$$\omega_0^2 R_{\mathrm{E}} = \frac{GM_{\mathrm{E}}}{R_{\mathrm{E}}^2} = g \qquad \text{and} \qquad \tau_0 = 2\pi\left(\frac{R_{\mathrm{E}}}{g}\right)^{1/2}. \tag{2.32}$$

After one year (i.e. after $\frac{t}{\tau_0}$ orbits with $t = 1$ year), we find (assuming $r \simeq R_{\mathrm{E}}$)

$$\Delta\alpha(t) = \Delta\alpha\frac{t}{\tau_0} = \frac{3\pi g R_{\mathrm{E}}}{c^2}\frac{t}{2\pi\sqrt{\frac{R_{\mathrm{E}}}{g}}} \simeq 8''.4\ \mathrm{yr}^{-1}. \tag{2.33}$$

For a general radius r, one finds $\Delta\alpha(t) \simeq 8''.4\,(R_{\mathrm{E}}/r)^{5/2}\ \mathrm{yr}^{-1}$.

On April 20, 2004 the satellite *Gravity Probe B* has been launched to measure the geodetic precession. At an altitude of 642 km, general relativity predicts a geodetic precession of 6606 mas yr^{-1}. The measured result was 6673 ± 97 mas yr^{-1} (1 mas = 1 milliarcsecond). This result therefore matches the predictions reasonably well.[4]

[4] C. Everitt et al., Classical and Quantum Gravity **25**, (2008) 114002

2.1.1 De Sitter Precession of the Moon

The Earth-Moon system can be considered as a "gyroscope" with an axis "perpendicular" to the orbital plane in an orbit around the Sun. We denote by $\boldsymbol{l}$ the angular momentum of the Earth-Moon system with respect to the common center of mass ($\boldsymbol{l}$ is essentially the orbital angular momentum of the Moon because the common center of mass almost coincides with the center of the Earth).

The angular momentum $\boldsymbol{l}$ can be decomposed in a component $\boldsymbol{l}_\perp$ perpendicular to the Earth's orbital plane around the Sun and a parallel component $\boldsymbol{l}_{||}$. The parallel component does not vanish because the orbital plane of the Moon around the Earth is tilted by 5° with respect to the orbital plane of the Earth around the Sun. In terms of the previously defined quantities, $\boldsymbol{l}_\perp$ corresponds to S^2 and stays constant. The component $\boldsymbol{l}_{||}$ corresponds to $\boldsymbol{S}$ and lies in the orbital plane of the Earth around the Sun. It is this parallel component which is affected by geodetic precession. Therefore the orbital plane of the Moon rotates slightly (this was first noticed by De Sitter in 1916).

We can calculate this precession per century

$$\Delta\alpha_{\text{De Sitter}} = 100 \frac{3\pi a_\odot}{r_{\text{Earth-Sun}}} \simeq 2'' \text{ per century} \tag{2.34}$$

where $a_\odot = 1.5$ km and $r_{\text{Earth-Sun}} = 1$ AU $\approx 150 \times 10^6$ km. Additionally we have a Newtonian precession (as in any three body system) with a period of 18.6 yr. This Newtonian effect is 10^7 times larger than the De Sitter precession. Nevertheless the De Sitter precession has been measured. A laser beam has been sent to the the Moon where it was reflected back to Earth by mirrors previously brought to the Moon by the Apollo mission (1969 and following years). The measurements that have been performed from 1970 till 1986 confirmed the De Sitter precession with a precision of about 1%.[5]

[5] Shapiro et al., Phys. Rev. Lett. **61**, 2643 (1988) and Müller et al., Astrophys. J. **382**, L101 (1991)

3 Linearized Field Equations

In order to find solutions to the Einstein field equations in the weak field regime, one can linearize the equations. The results of this section will, for example, be used to calculate the Lense-Thirring effect and to describe gravitational waves.

Since the field equations are non-linear, there is no standard procedure to solve the equations given a source for the fields $T_{\mu\nu}$ (for example, in electrodynamics we didn't have to struggle with such difficulties). Besides numerical methods, there are essentially three possibilities as for which type of solutions one can find:

- **Exact solutions** assuming simplifying conditions (as, for example, staticity, isotropy, ...). An example for this case is the Schwarzschild solution.
- Solutions of the **linearized field equations** for weak gravitational fields.
- Systematic expansion of the field equations and the equations of motion for weak fields and small velocities. This method is also called **post-Newtonian approximation**. For example, in planetary systems we have $\frac{v^2}{c^2} \sim \frac{\phi}{c^2}$ which is very small. Such results should reproduce the Newtonian limit in lowest order (linearized in ϕ).

We shall now elaborate the second possibility.

3.1 The Energy Momentum Tensor of the Gravitational Field

The field itself is a form of energy and thus also a source of the field. This effect is purely due to the non-linearities, of course. Considering a weak field, we can work with small deviations from the Minkowski metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \qquad \text{with } |h_{\mu\nu}| \ll 1. \tag{3.1}$$

One proceeds as follows. First, $G_{\mu\nu}$ has to be expanded in powers of $h_{\mu\nu}$. The first order terms will lead to a linear wave equation. Neglecting terms of third order, the second order terms give the energy-momentum tensor of the gravitational field.

The expansion of the Ricci tensor can be written as

$$R_{\mu\nu} = R^{(1)}_{\mu\nu} + R^{(2)}_{\mu\nu} + ... \tag{3.2}$$

with $R^{(0)}_{\mu\nu} = 0$. In order to compute the first order term of (3.2), we write down the expansion of the curvature tensor: [6]

$$R_{\rho\mu\sigma\nu} = \frac{1}{2}\left(g_{\rho\sigma,\mu,\nu} + g_{\mu\nu,\rho,\sigma} - g_{\mu\sigma,\nu,\rho} - g_{\rho\nu,\sigma,\mu}\right) + \mathcal{O}(h^2) \tag{3.3}$$

[6] Notice that we adopt here a different notation with respect to GR I: we adopt the "minus" convention of the EFE (i.e. $R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} = -\frac{8\pi G}{c^4}T_{\mu\nu}$), which simply follows from a change of sign of the Riemann tensor. The Riemann tensor from which (3.3) is derived differs thus from (GR I, 17.4) by a factor -1.

where the derivatives are non-covariant (the additional terms due to covariant derivatives are of higher order). We can thus write the first order Ricci tensor in terms of $h_{\mu\nu}$:

$$R^{(1)}_{\mu\nu} = \frac{1}{2}\left(\Box h_{\mu\nu} + h^{\rho}{}_{\rho,\mu,\nu} - h^{\rho}{}_{\mu,\rho,\nu} - h^{\rho}{}_{\nu,\rho,\mu}\right). \tag{3.4}$$

The d'Alembert operator can be used instead of $\partial_\mu \partial^\mu$ because in the approximation (3.1) the coordinates are "almost" Minkowskian, so $\partial_\mu \partial^\mu = \Box + \mathcal{O}(h)$. The first order Ricci scalar is given by

$$R^{(1)} = \eta^{\lambda\rho} R^{(1)}_{\lambda\rho}. \tag{3.5}$$

We proceed by considering the second order equations. The left-hand side of the field equations can be written in terms of the quantity $t_{\mu\nu}$ which is defined by

$$R^{(2)}_{\mu\nu} - \left(\frac{R g_{\mu\nu}}{2}\right)^{(2)} =: \frac{8\pi G}{c^4} t_{\mu\nu}. \tag{3.6}$$

We take these terms to the right-hand side of Einstein's equations and find at second order in $h_{\mu\nu}$:

$$R^{(1)}_{\mu\nu} - \frac{R^{(1)}}{2}\eta_{\mu\nu} = -\frac{8\pi G}{c^4}(T_{\mu\nu} + t_{\mu\nu}). \tag{3.7}$$

This can be interpreted as a wave equation linear in $h_{\mu\nu}$ with source terms

$$\tau_{\mu\nu} = T_{\mu\nu} + t_{\mu\nu}. \tag{3.8}$$

We have to think of $\tau_{\mu\nu}$ as being the energy-momentum tensor which also includes the contribution of the gravitational field itself.

We interpret (3.7) as follows: since $G^{\mu\nu}{}_{;\nu} = 0$ (Bianchi identity), we find for the left-hand side of (3.7):

$$\frac{\partial}{\partial x_\nu}\left(R^{(1)}_{\mu\nu} - \frac{R^{(1)}}{2}\eta_{\mu\nu}\right) = 0. \tag{3.9}$$

Therefore the right-hand side satisfies

$$\frac{\partial \tau_{\mu\nu}}{\partial x_\nu} = 0. \tag{3.10}$$

This gives the momentum

$$P_\mu = \int d^3r\ \tau_{\mu 0} = \text{const.} \tag{3.11}$$

which is conserved (in time). We can thus interpret $\tau_{\mu 0}$ as the momentum density and $\tau_{\mu\nu}$ as an energy-momentum tensor (indeed we know that $T^{\mu\nu}{}_{;\nu} = 0$ but so far we did not necessarily conclude $\tau^{\mu\nu}{}_{;\nu} = 0$). Since $T_{\mu\nu}$ includes all non-gravitational sources and $\tau_{\mu\nu}$ is interpreted as the "complete" energy-momentum tensor, $t_{\mu\nu}$ clearly describes energy-momentum which is purely due to the gravitational field:

$$t^{\text{grav.}}_{\mu\nu} = \frac{c^4}{8\pi G}\left(R^{(2)}_{\mu\nu} - \left(\frac{R g_{\mu\nu}}{2}\right)^{(2)}\right) \qquad (|h_{\mu\nu}| \ll 1). \tag{3.12}$$

3.2 Linearized Field Equations

With Eq. (3.3) we find for the field equations at first order in h

$$\Box h_{\mu\nu} + h^{\rho}{}_{\rho,\mu,\nu} - h^{\rho}{}_{\mu,\rho,\nu} - h^{\rho}{}_{\nu,\rho,\mu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}\right). \tag{3.13}$$

We use $\eta_{\mu\nu}$ instead of $g_{\mu\nu}$ in this equation because both sides are already of order h. Since the field equations are covariant, we are free to perform a coordinate transformation. But note that since $|h_{\mu\nu}| \ll 1$ we can only perform coordinate transformations which deviate only slightly from Minkowski coordinates:

$$x^{\mu} \longrightarrow x'^{\mu} = x^{\mu} + \varepsilon^{\mu}(x) \qquad \text{with } \varepsilon \ll 1. \tag{3.14}$$

From $g'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\rho}} g^{\lambda\rho}$ we infer how $h_{\mu\nu}$ transforms. With $\frac{\partial x'^{\mu}}{\partial x^{\lambda}} = \delta^{\mu}_{\lambda} + \frac{\partial \varepsilon^{\mu}}{\partial x^{\lambda}}$ inserted into $g'^{\mu\nu}$ we get

$$\begin{aligned} g'^{\mu\nu} &= \eta^{\mu\nu} - h'^{\mu\nu} \\ &= \left(\delta^{\mu}_{\lambda} + \frac{\partial \varepsilon^{\mu}}{\partial x^{\lambda}}\right)\left(\delta^{\nu}_{\rho} + \frac{\partial \varepsilon^{\nu}}{\partial x^{\rho}}\right)\left(\eta^{\lambda\rho} - h^{\lambda\rho}\right) \end{aligned} \tag{3.15}$$

where we used that from $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ it follows $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$. From Eq. (3.15) we infer

$$h'^{\mu\nu} = h^{\mu\nu} - \frac{\partial \varepsilon^{\mu}}{\partial x_{\nu}} - \frac{\partial \varepsilon^{\nu}}{\partial x_{\mu}}. \tag{3.16}$$

Since this is already a first order equation (in h), we can raise and lower indices with $g_{\mu\nu} \simeq \eta_{\mu\nu}$ and $g^{\mu\nu} \simeq \eta^{\mu\nu}$. Thus

$$h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \varepsilon_{\mu}}{\partial x^{\nu}} - \frac{\partial \varepsilon_{\nu}}{\partial x^{\mu}}. \tag{3.17}$$

In analogy to electrodynamics this transformation of the "potentials" $g_{\mu\nu}$ is called a **gauge transformation**. We can choose four functions $\varepsilon^{\mu}(x)$ which give four constraints on the "potentials" $h_{\mu\nu}$. For instance,

$$2h^{\mu}{}_{\nu,\mu} = h^{\mu}{}_{\mu,\nu}. \tag{3.18}$$

We insert the gauge condition (3.18) into (3.13) and obtain the **decoupled linearized field equations**:

$$\boxed{\Box h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}\right)}. \tag{3.19}$$

This can easily be seen if we differentiate (3.18) (i.e. $h^{\rho}{}_{\rho,\mu} = 2h^{\rho}{}_{\mu,\rho}$) with respect to x^{ν}:

$$h^{\rho}{}_{\rho,\mu,\nu} = 2h^{\rho}{}_{\mu,\rho,\nu} = h^{\rho}{}_{\mu,\rho,\nu} + h^{\rho}{}_{\nu,\rho,\mu} \tag{3.20}$$

(we used $h_{\mu\nu} = h_{\nu\mu}$). This implies

$$-h^{\rho}{}_{\mu,\rho,\nu} - h^{\rho}{}_{\nu,\rho,\mu} + h^{\rho}{}_{\rho,\mu,\nu} = 0\ , \tag{3.21}$$

which is just another form of our gauge condition from which it can be seen that (3.13) indeed reduces to (3.19).

Furthermore, it can be shown that from (3.17) it follows that if $h_{\mu\nu}$ does not satisfy (3.18), then we can find a transformed $h'_{\mu\nu}$ that does so. This can be done using the coordinate transformation (3.14) with $\Box\varepsilon_\nu = h^\mu{}_{\nu,\mu} - \frac{1}{2}h^\mu{}_{\mu,\nu}$.

The linearized field equation Eq. (3.19) has the same structure as the field equations in electrodynamics. We can therefore immediately write down the well-known solution for the retarded potentials:

$$h_{\mu\nu}(\boldsymbol{r},t) = -\frac{4G}{c^4}\int d^3r' \, \frac{S_{\mu\nu}\left(\boldsymbol{r'}, t - \frac{|\boldsymbol{r}-\boldsymbol{r'}|}{c}\right)}{|\boldsymbol{r}-\boldsymbol{r'}|} \tag{3.22}$$

$$\text{with} \qquad S_{\mu\nu} = T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}.$$

The interpretation is the same as in electrodynamics: a change in $S_{\mu\nu}$ at podition $\boldsymbol{r'}$ does not affect the position $\boldsymbol{r}$ before some time $\frac{|\boldsymbol{r}-\boldsymbol{r'}|}{c}$ has passed.

4 Lense-Thirring Effect

The Lense-Thirring effect is the precession of a gyroscope in the gravitational field of the Earth due to the Earth's rotation. To set up an analogy with electrodynamics, we note that the gravitational field of the Schwarzschild metric corresponds to the Coulomb field outside of a static, spherical charge distribution. If a charge distribution rotates with constant angular velocity, this results in the presence of a static, non-isotropic magnetic field. Similarly, the rotation of the Earth will cause a *gravitomagnetic* field.

We will treat this problem by using the linearized field equations. Another approach would be to start from the exact Kerr solution (i.e. the metric outside of a rotating black hole) and apply the weak field limit.

4.1 Metric of the rotating Earth

We assume a weak field caused by a slowly rotating planet, so $|h_{\mu\nu}| \ll 1$. The linearized field equations read

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}\right). \tag{4.1}$$

The coordinates $x^\mu = (x^0, ..., x^3)$ are Minkowski coordinates up to corrections of $\mathcal{O}(h)$. In the energy-momentum tensor (19.9) we can neglect the pressure since $p \ll \rho c^2$. Since the velocities (rotation of the Earth) are small compared to c, we neglect terms of order $\left(\frac{v}{c}\right)^2$. The energy-momentum tensor thus reads

$$T_{\mu\nu} \simeq \rho c^2 \begin{pmatrix} 1 & \frac{v_i}{c} \\ \frac{v_i}{c} & 0 \end{pmatrix}. \tag{4.2}$$

The terms proportional to v_i generate the gravitomagnetic field. This has to be compared to electrodynamics where magnetic fields are generated by currents.

The mass distribution of the Earth can be approximated as follows:

$$\rho(\boldsymbol{r}) = \begin{cases} \rho_0 & (r \leq R_{\mathrm{E}}) \\ 0 & (r > R_{\mathrm{E}}) \end{cases} \tag{4.3}$$

The angular velocity of the Earth is

$$\boldsymbol{\omega} = \omega \boldsymbol{e}_3 \quad \text{with } \omega = \frac{2\pi}{1 \text{ day}}. \tag{4.4}$$

We consider the Earth as a rigid body, so we can write the velocity field as

$$\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{\omega} \wedge \boldsymbol{r} \quad \text{or } v_i = \varepsilon_{ijk}\omega^j r^k. \tag{4.5}$$

Because this velocity is constant, $T_{\mu\nu}$ does not depend on time and therefore the field equations (4.1) have stationary solutions. We can thus replace $\Box$ by $-\Delta$ and Eq. (4.1) becomes

$$\Delta h_{\mu\mu} = \frac{8\pi G}{c^2}\rho(\boldsymbol{r}), \tag{4.6}$$

$$\Delta h_{0i} = \frac{16\pi G}{c^3}\rho(\boldsymbol{r})\varepsilon_{ijk}\omega^j r^k \tag{4.7}$$

where we used $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, $\frac{T}{2} = \frac{\rho c^2}{2}$ and $T_{00} = \rho c^2$, $T_{ii} = 0$.

Using

$$\Delta \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} = -4\pi \delta^{(3)}(\boldsymbol{r} - \boldsymbol{r}'), \tag{4.8}$$

we can immediately solve the equations (4.6) and (4.7):

$$h_{\mu\mu}(\boldsymbol{r}) = -\frac{2G}{c^2} \int d^3r' \, \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|}, \tag{4.9}$$

$$h_{0i}(\boldsymbol{r}) = -\frac{4G}{c^3} \varepsilon_{ikn} \omega^k \int d^3r' \, \frac{\rho(\boldsymbol{r}') x'^n}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{4.10}$$

where x'^n denotes the n-th component of $\boldsymbol{r}'$.

For the region outside of the mass distribution ($r > R_{\mathrm{E}}$), we can use the expansion

$$\begin{aligned} \frac{1}{|\boldsymbol{r} - \boldsymbol{r}'|} &= \sum_{l,m} \frac{4\pi}{(2l+1)} \frac{r'^l}{r^{l+1}} Y^*_{lm}(\hat{\boldsymbol{r}}') Y_{lm}(\hat{\boldsymbol{r}}) \qquad (r > r') \\ &= \frac{1}{r} - \frac{x^j x'_j}{r^3} + \dots \end{aligned} \tag{4.11}$$

where Y_{lm} are the spherical harmonic functions and $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{r}}'$ denote unit vectors in directions of $\boldsymbol{r}$ and $\boldsymbol{r}'$, respectively. Note that for the Cartesian components we have $x_i = g_{ik} x^k = -x^i + \mathcal{O}(h)$. The corrections of $\mathcal{O}(h)$ can thus be neglected because the right-hand sides of (4.9) and (4.10) are already of first order in h. Since $\rho(\boldsymbol{r})$ is spherically symmetric, only the first term of (4.11) contributes in Eq. (4.9):

$$h_{\mu\mu}(\boldsymbol{r}) = -\frac{2G}{c^2 r} \int d^3r' \, \rho(\boldsymbol{r}') = -\frac{2GM_{\mathrm{E}}}{c^2 r} \quad (r \geq R_{\mathrm{E}}) \; . \tag{4.12}$$

Since $\rho x'^n$ is proportional to Y_{1m}, only terms with $l = 1$ contribute to (4.10):

$$\begin{aligned} h_{0i}(\boldsymbol{r}) &= \frac{4G}{c^3} \varepsilon_{ijn} \frac{\omega^j x^k}{r^3} \int d^3r' \, \rho(\boldsymbol{r}') x'^n x'_k \\ &= -\frac{4GM_{\mathrm{E}} R_{\mathrm{E}}^2}{5c^3} \varepsilon_{ijn} \frac{\omega^j x^n}{r^3} \qquad (r \geq R_{\mathrm{E}}). \end{aligned} \tag{4.13}$$

Since $\rho(\boldsymbol{r})$ is spherically symmetric, we integrated using $x'^n x'_k = -\delta^n_k \frac{r'^2}{3}$ and furthermore used $\rho_0 = 3M_{\mathrm{E}}/(4\pi R_{\mathrm{E}}^3)$. Considering h_{0i} as a vector, i.e. $h_{0i} \to \boldsymbol{h} = h_{0i} \boldsymbol{e}^i$, we can write (4.7) as

$$\Delta \boldsymbol{h}(\boldsymbol{r}) = \frac{16\pi G}{c^3} \rho \boldsymbol{\omega} \wedge \boldsymbol{r} \tag{4.14}$$

and (4.13) becomes

$$\boxed{\boldsymbol{h}(\boldsymbol{r}) = -\frac{4GM_{\mathrm{E}} R_{\mathrm{E}}^2}{5c^3} \frac{\boldsymbol{\omega} \wedge \boldsymbol{r}}{r^3} = -\frac{2GI}{c^3} \frac{\boldsymbol{\omega} \wedge \boldsymbol{r}}{r^3}} \tag{4.15}$$

where $I = \frac{2}{5} M_{\mathrm{E}} R_{\mathrm{E}}^2$ is the moment of inertia of a homogeneous sphere.

We want to consider the analogy with electrodynamics again. In magnetostatics, the vector potential $\boldsymbol{A}$ of a homogeneously charged rotating sphere with radius R and total charge q satisfies

$$\Delta \boldsymbol{A} = -\frac{4\pi}{c}\rho_{\mathrm{e}}\boldsymbol{\omega} \wedge \boldsymbol{r} \tag{4.16}$$

$$\Rightarrow \quad \boldsymbol{A} = \frac{qR^2}{5c}\frac{\boldsymbol{\omega} \wedge \boldsymbol{r}}{r^3} \tag{4.17}$$

which has the same form as Eq. (4.15).

Eqs. (4.12) and (4.13) determine the metric of the rotating Earth (valid for $r \geq R_{\mathrm{E}}$):

$$\boxed{ds^2 = \left(1 - \frac{2GM_{\mathrm{E}}}{c^2 r}\right)c^2 dt^2 - \left(1 + \frac{2GM_{\mathrm{E}}}{c^2 r}\right)d\boldsymbol{r}^2 + 2ch_{0i}\, dx^i dt} \tag{4.18}$$

where $d\boldsymbol{r}^2 = -dx^i dx_i$. Note that this metric at $\mathcal{O}\left(\frac{GM_{\mathrm{E}}}{c^2 r}\right)$ and for $\omega = 0$ does not reduce to the Schwarzschild metric since Eq. (4.1) implies that we chose other coordinates as compared to the standard form. However, the metric (4.18) asymptotically ($r \to \infty$) becomes the Minkowski metric. One can perform a coordinate transformation such that $d\boldsymbol{r}^2$ has the usual angular dependence, i.e. $d\boldsymbol{r}^2 \to dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2)$. Distant "fixed" stars (which live in the asymptotic Minkowski spacetime) have constant values for the angles (θ, φ). Therefore changes in the angles due to spin precession refer to rotations with respect to the distant fixed stars.

4.1.1 Rotation of the Local IS

The metric (4.18) implies a rotation of the local inertial system IS. In order to see this, consider the axis of a gyroscope which is described by

$$\frac{dS^\mu}{d\tau} = -\Gamma^\mu{}_{\kappa\nu}S^\kappa u^\nu. \tag{4.19}$$

A freely falling gyroscope would not only show a precession linked to the Lense-Thirring effect (that can be loosely defined as "precession due to the rotation of the central body"), but would also be subjected, because of its motion in a curved metric, to the geodesic precession derived in Section 2. However, we want here to artificially separate these two different contributions, and only look at the pure Lense-Thirring component. This can be done with a simple trick: we consider the gyroscope as being at rest in the external frame, and just let it free to change its rotational axis. Thus we have, in the coordinate system defined by Eq. (4.18),

$$S^\mu = (0, S^i), \qquad u^\mu = (c, \mathbf{0}). \tag{4.20}$$

Because we are taking into account only terms up to $\mathcal{O}(h)$, we can replace $\frac{dS^i}{d\tau}$ by $\frac{dS^i}{dt}$. From Eqs. (4.19) and (4.20) it follows that

$$\frac{dS^i}{dt} = -c\Gamma^i{}_{0j}S^j. \tag{4.21}$$

For time independent $h_{\mu\nu}$ we find at first order

$$\Gamma^i{}_{0j} = \frac{\eta^{ik}}{2}\left(\frac{\partial h_{0k}}{\partial x^j} - \frac{\partial h_{0j}}{\partial x^k}\right) = \frac{1}{2}\left(\partial_j h_0{}^i - \partial^i h_{0j}\right). \tag{4.22}$$

Inserting this into (4.21) and lowering the index i, we find

$$\begin{aligned}\frac{dS_i}{dt} &= -\frac{c}{2}\left(\partial_j h_{0i} - \partial_i h_{0j}\right) S^j \\ &= -\frac{c}{2}\varepsilon_{ikl}\left(\varepsilon^{kmn}\partial_m h_{0n}\right) S^l \\ &= \varepsilon_{ikl}\Omega^k S^l \end{aligned} \tag{4.23}$$

with $\varepsilon_{ikl}\varepsilon^{kmn} = \delta_l^m\delta_i^n - \delta_l^n\delta_i^m$ and the **gravitomagnetic field**

$$\Omega^k = -\frac{c}{2}\varepsilon^{kmn}\partial_m h_{0n} \quad \text{or} \quad \boldsymbol{\Omega}(\boldsymbol{r}) = -\frac{c}{2}\boldsymbol{\nabla}\wedge\boldsymbol{h}(\boldsymbol{r}) \ . \tag{4.24}$$

The indices i, j, m run over $1, 2, 3$. We call $\boldsymbol{h}$ the **gravitomagnetic potential**. Using $\boldsymbol{h}$ from Eq. (4.15) we obtain the following expression for the angular velocity in the local IS:

$$\boxed{\boldsymbol{\Omega}(\boldsymbol{r}) = \frac{2GM_{\mathrm{E}}R_{\mathrm{E}}^2}{5c^2}\frac{3(\boldsymbol{\omega}\cdot\boldsymbol{r})\boldsymbol{r} - \boldsymbol{\omega}r^2}{r^5}}. \tag{4.25}$$

Note that $\boldsymbol{\Omega}$ has the same form as $\boldsymbol{B} = \boldsymbol{\nabla}\wedge\boldsymbol{A}$ in electrodynamics. In vector form, Eq. (4.23) reads

$$\boxed{\frac{d\boldsymbol{S}}{dt} = \boldsymbol{\Omega}\wedge\boldsymbol{S} \quad \text{or} \quad d\boldsymbol{S} = (\boldsymbol{\Omega}dt)\wedge\boldsymbol{S}}. \tag{4.26}$$

This implies a precession of the spin of the gyroscope's axis with angular velocity $\boldsymbol{\Omega}$. This precession is due to the rotation of the Earth with angular velocity $\boldsymbol{\omega}$. The precession of the gyroscope's axis is equivalent to the rotation of the local IS because in the local IS we have $\boldsymbol{S}$ =const.. Therefore the local IS rotates with angular velocity $\boldsymbol{\Omega}$ as compared to the global coordinate system described by (4.18). For $r \to \infty$ Eq. (4.18) becomes Minkowskian, i.e. it describes a system which doesn't rotate with respect to the fixed star system.

To summarize, the physical meaning of the angular velocity $\boldsymbol{\Omega}$ is that the local IS rotates with Ω with respect to the fixed star system. The rotation of the Earth drags the local IS ("**frame dragging**").

Now that we derived geodetic precession and the Lense-Thirring effect, we note that in fact both effects take place at the same time and thus sum up. Inserting $r = R_{\mathrm{E}}$ in Eq. (4.25) we get at the north pole and at the equator, respectively:

$$\boldsymbol{\Omega} = \frac{2GM_{\mathrm{E}}}{5c^2R_{\mathrm{E}}}\boldsymbol{\omega}\cdot\begin{cases} 2 & \text{(North pole, } \theta = 0) \\ (-1) & \text{(equator, } \theta = \frac{\pi}{2}) \end{cases} \tag{4.27}$$

where we used $|\boldsymbol{\omega}\cdot\boldsymbol{r}| = \omega r\cos\theta$. This evaluates to

$$\frac{2GM_{\mathrm{E}}}{5c^2R_{\mathrm{E}}}\omega \sim 10^{-9}\omega. \tag{4.28}$$

We conclude that the Lense-Thirring precession affects a Foucault pendulum located at the north pole. The rotation with respect to the distant stars amounts to

$$\Delta\phi = \Omega_{\mathrm{LT}}\cdot 1\mathrm{yr} = \frac{4GM_{\mathrm{E}}}{5c^2R_{\mathrm{E}}}2\pi\cdot 365 = 0.2'' \text{ per year.} \tag{4.29}$$

This effect was first computed by Lense and Thirring in 1918.[7]

The NASA satellite *Gravity Probe B* aims to measure both geodetic precession and Lense-Thirring precession. It was launched in 2004 and orbits the Earth on a polar orbit. If one chooses the spin perpendicular to the orbital plane, the Lense-Thirring effect vanishes for equatorial orbits since in this case we have $\boldsymbol{S}||\boldsymbol{w}||\boldsymbol{\Omega}$ and Eq. (4.26) gives $\frac{d\boldsymbol{S}}{dt} = 0$. The expected total frame dragging is $0''.05$ per year and it is yet unclear whether the instrumental precession suffices to detect the effect. However there are claims that the effect has been measured on the orbit of the LAGEOS satellites.[8]

Update: in May 2011 the final results of Gravity Probe B have been released and they indeed confirm the GR predictions to a high accuracy.[9]

4.2 Gravitomagnetic Forces

In the discussion of the Lense-Thirring precession, we saw a formal equivalence to electromagnetism:

$$\boldsymbol{h} \quad \leftrightarrow \quad \boldsymbol{A}$$

$$\boldsymbol{\Omega} \quad \leftrightarrow \quad \boldsymbol{B}.$$

This analogy persists even for the equation of motion of a particle in the metric (4.18),

$$\frac{du^\mu}{d\tau} = -\Gamma^\mu{}_{\gamma\nu} u^\gamma u^\nu. \tag{4.30}$$

Neglecting terms $\mathcal{O}\left(v^2/c^2\right)$ we have $d\tau \approx dt$ and $u^\mu = (c, v^i)$. Therefore Eq. (4.30) reads

$$\frac{dv^i}{dt} = -\Gamma^i{}_{00}c^2 - 2c\Gamma^i{}_{0j}v^j + \mathcal{O}\left(\frac{v^2}{c^2}\right). \tag{4.31}$$

The first term on the right-hand side corresponds to the gradient of the Newtonian potential. In analogy to to (4.21)-(4.23), the second term can be shown to give

$$-\Gamma^i{}_{0j}v^j = (\boldsymbol{\Omega} \wedge \boldsymbol{v})^i. \tag{4.32}$$

This implies that Eq. (4.31) can be written as

$$\frac{d\boldsymbol{v}}{dt} = -\mathrm{grad}\phi + 2\boldsymbol{\Omega} \wedge \boldsymbol{v}. \tag{4.33}$$

This is the equation of motion in the presence of gravitomagnetic forces and it has the same structure as the equation for the Lorentz force

$$\boldsymbol{K} = q\left(\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \wedge \boldsymbol{B}\right). \tag{4.34}$$

This analogy is the origin of the notion of **gravitomagnetism**.

Note, however, that this analogy is only true if we consider the linearized field equations of general relativity. In the above identification the electromagnetic fields are full solutions to Maxwell's equations whereas the gravitomagnetic potential and field are approximations. Furthermore, the analogy is quite formal and certainly not complete due to the absence of negative "gravitational charges".

[7] Lense and Thirring, Phys. Zeitschr. **19**, 156 (1918)

[8] I. Ciufuloni and Pavlis, Nature **431**, (2004) 958

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5 Gravitational Waves

For weak gravitational fields (i.e. $|h_{\mu\nu}| = |g_{\mu\nu} - \eta_{\mu\nu}| \ll 1$) the Einstein field equations read

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}\right). \tag{5.1}$$

In the vacuum ($T_{\mu\nu} = 0$) the equation reduces to

$$\Box h_{\mu\nu} = 0 \tag{5.2}$$

which has plane waves as its simplest solution. The above equation is quite similar to the wave equation in electromagnetism, $\Box A^\mu = 0$ with the electromagnetic vector potential A^μ. As we will see, the solutions are similar, as well. Note that the wave equation in electromagnetism is exact whereas the general relativistic wave equation arises from the approximate linearized field equations.

5.1 Electromagnetic Waves

Physical fields are invariant under gauge transformations

$$A^\mu \to A'^\mu = A^\mu + \partial^\mu \chi \tag{5.3}$$

so that we can choose $\partial_\mu A^\mu = 0$ (Lorenz gauge) and get

$$\Box A^\mu = \frac{4\pi}{c} j^\mu. \tag{5.4}$$

Due to the gauge conditions, only three out of four components of A^μ are independent. While leaving the Lorenz gauge unaltered, we still have the freedom to perform an additional gauge transformation satisfying $\Box\chi = 0$. Since in vacuum $j^\mu = 0$, this allows us to set $A^0 = 0$. Finally we are left with two degrees of freedom (polarizations). The conditions read then

$$\Box A^\mu = 0, \qquad A^0 = 0, \qquad \partial_i A^i = 0. \tag{5.5}$$

This is solved by the ansatz

$$A^\mu = e^\mu \exp[-ik_\nu x^\nu] + \text{c.c.}\ , \tag{5.6}$$

where $k_\mu k^\mu = 0$ and $e_i k^i = 0$ (polarizations are transverse to propagation direction).

5.2 The Case of Gravity

Due to the symmetry $h_{\mu\nu} = h_{\nu\mu}$, 10 out of 16 components of $h_{\mu\nu}$ are independent. With a gauge transformation of the form (3.18) we can impose four additional conditions. This leaves us with 6 degrees of freedom that are truly independent. If we consider the vacuum case

$$\Box h_{\mu\nu} = 0, \tag{5.7}$$

in addition to (3.17) we can perform a further transformation of the form

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \varepsilon_\nu - \partial_\nu \varepsilon_\mu \tag{5.8}$$

provided that ε_μ satisfies

$$\Box\varepsilon_\mu = 0. \tag{5.9}$$

Such a transformation leaves Eq. (5.7) and the gauge condition (3.18) invariant (this is in complete analogy to electromagnetism, of course). With these four additional conditions we are left with two independent components of $h_{\mu\nu}$. The solution to (5.7) can be written in terms of plane waves

$$h_{\mu\nu} = e_{\mu\nu} \exp[-ik_\kappa x^\kappa] + \text{c.c.} \tag{5.10}$$

where

$$\eta^{\lambda\nu} k_\lambda k_\nu = k^\nu k_\nu = 0 \quad \Leftrightarrow \quad k_0^2 = \frac{\omega^2}{c^2} = \boldsymbol{k}^2 = k^2. \tag{5.11}$$

The amplitude of the wave $e_{\mu\nu}$ is called **polarization tensor**. Inserting (5.10) into the gauge condition (3.18) $(2h^\mu{}_{\nu,\mu} = h^\mu{}_{\mu,\nu})$ leads to

$$2k_\mu \eta^{\mu\rho} e_{\rho\nu} = k_\nu \eta^{\mu\rho} e_{\rho\mu}. \tag{5.12}$$

Clearly $e_{\mu\nu}$ inherits the symmetry of $h_{\mu\nu}$, thus $e_{\mu\nu} = e_{\nu\mu}$. Let us choose a wave travelling along the x^3-axis. This yields the wave solution

$$h_{\mu\nu} = e_{\mu\nu} \exp\left[ik(x^3 - ct)\right] \tag{5.13}$$

where we used Eq. (5.11). The components of the wave vector are then

$$k_1 = k_2 = 0, \qquad k_0 = -k_3 = k = \frac{\omega}{c}. \tag{5.14}$$

In this case the gauge condition (5.12) reads

$$e_{00} + e_{30} = \frac{1}{2}(e_{00} - e_{11} - e_{22} - e_{33}), \tag{5.15}$$

$$e_{01} + e_{31} = 0, \tag{5.16}$$

$$e_{02} + e_{32} = 0, \tag{5.17}$$

$$e_{03} + e_{33} = -\frac{1}{2}(e_{00} - e_{11} - e_{22} - e_{33}). \tag{5.18}$$

With $e_{\mu\nu} = e_{\nu\mu}$ and these four conditions, the polarization tensor is fully determined by six components. All the other components can be expressed in terms of the six independent components

$$e_{00},\ e_{11},\ e_{33},\ e_{12},\ e_{13} \text{ and } e_{23}. \tag{5.19}$$

The other components are given by

$$e_{01} = -e_{31} = -e_{13}, \quad e_{02} = -e_{32}, \quad e_{22} = -e_{11}, \quad e_{03} = -\frac{1}{2}(e_{00} + e_{33}). \tag{5.20}$$

We can perform yet another transformation (3.14) ($x'^\mu = x^\mu + \varepsilon^\mu$) with functions ε^μ satisfying $\Box\varepsilon^\mu = 0$. The functions are solutions of the wave equation, therefore we can write them as

$$\varepsilon^\mu(x) = \delta^\mu \exp\left[-ik_\mu x^\mu\right] + \text{c.c.}. \tag{5.21}$$

As noted before, such a transformation with arbitrary δ^μ does not violate the gauge condition (3.18). We choose k^μ in (5.21) equal to the wave vector of a given gravitational wave. Using (5.21) in (3.16) we obtain a new solution $h'_{\mu\nu}$ in which all the terms have the same exponential dependence of $\exp[-ik_\mu x^\mu]$. Thus only the amplitudes transform as

$$e'_{11} = e_{11}, \tag{5.22}$$

$$e'_{12} = e_{12}, \tag{5.23}$$

$$e'_{13} = e_{13} - i\delta_1 k, \tag{5.24}$$

$$e'_{23} = e_{23} - i\delta_2 k, \tag{5.25}$$

$$e'_{33} = e_{33} - 2i\delta_3 k, \tag{5.26}$$

$$e'_{00} = e_{00} + 2ik\delta_0. \tag{5.27}$$

We can choose δ_μ such that $e'_{00} = e'_{13} = e'_{33} = e'_{23} = 0$. This new solution is equivalent to the old one. From the physical point of view, only polarizations corresponding to e'_{11} and e'_{12} are relevant.

Neglecting primes in our notation from now on, we get for the gravitational wave propagating in x^3-direction, after gauging away all redundancies

$$\boxed{h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{11} & e_{12} & 0 \\ 0 & e_{12} & -e_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cdot \exp\left[ik(x^3 - ct)\right] + \text{c.c.}} \tag{5.28}$$

5.2.1 Helicity

The direction of $\boldsymbol{k}$ is the x^3-axis. We ask now the question how (5.28) transforms under a rotation around this axis. Since we are in an almost Minkowskian metric we can realize this transformation as a Lorentz transformation described by the matrix

$$\bar{\Lambda}^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi & 0 \\ 0 & -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5.29}$$

Therefore the polarization tensor transforms as

$$e'_{\mu\nu} = \bar{\Lambda}^\rho{}_\mu \bar{\Lambda}^\sigma{}_\nu e_{\rho\sigma}. \tag{5.30}$$

This yields

$$e'_{11} = e_{11}\cos(2\varphi) + e_{12}\sin(2\varphi), \tag{5.31}$$

$$e'_{12} = -e_{11}\sin(2\varphi) + e_{12}\cos(2\varphi). \tag{5.32}$$

If we consider $e_\pm \equiv e_{11} \pm ie_{12}$ instead, we thus have

$$e'_\pm = e^{\pm 2i\varphi} e_\pm. \tag{5.33}$$

The vectors $e_\pm$ have **helicity** ± 2, whereas the wave solutions in electrodynamics have helicity ± 1. Generalizing from the electromagnetic field, which is quantized using a spin 1 particle, the photon, one can thus expect the quanta of the gravitational field to be spin 2 particles. While there is neither evidence for their existence nor a closed theory of quantum gravity, the hypothetical quanta of the gravitational field are commonly dubbed **gravitons** .

5.3 Particles in the Field of a Wave

Similarly to electromagnetic waves also gravitational waves exert forces on massive particles. We want to explore how the positions of particles are affected in the field of a gravitational wave. For this analyis consider a plane wave along the x^3-direction

$$h_{\mu\nu}(x^3, t) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{11} & e_{12} & 0 \\ 0 & e_{12} & -e_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \exp\left(ik(x^3 - ct)\right) + \text{c.c. }. \tag{5.34}$$

The corresponding metric has the form

$$ds^2 = \left(\eta_{\mu\nu} + h_{\mu\nu}(x^3, t)\right) dx^\mu dx^\nu. \tag{5.35}$$

The trajectory $x^\sigma(\tau)$ of a particle in the gravitational field satisfies the equation of motion

$$\frac{d^2x^\sigma}{d\tau^2} = -\Gamma^\sigma{}_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} \tag{5.36}$$

where $\Gamma^\sigma{}_{\mu\nu}$ can be taken from Eq. (5.28). We assume that there are no other forces but gravity which act on the particles. Inserting (5.34) into

$$\Gamma^\sigma{}_{\mu\nu} = \frac{1}{2}\eta^{\sigma\lambda}\left(\frac{\partial h_{\nu\lambda}}{\partial x^\mu} + \frac{\partial h_{\mu\lambda}}{\partial x^\nu} - \frac{\partial h_{\mu\nu}}{\partial x^\lambda}\right) + \mathcal{O}\left(h^2\right) \tag{5.37}$$

it follows

$$\Gamma^i{}_{00} = -\frac{1}{2}\left(\frac{\partial h_{0i}}{\partial x^0} + \frac{\partial h_{0i}}{\partial x^0} - \frac{\partial h_{00}}{\partial x^i}\right) = 0. \tag{5.38}$$

As initial conditions we choose

$$\dot{x}^i(0) = \left.\frac{dx^i}{d\tau}\right|_{\tau=0} = 0. \tag{5.39}$$

This implies

$$\left.\frac{d^2x^i}{d\tau}\right|_{\tau=0} = -\Gamma^i_{\mu\nu}\dot{x}^\mu(0)\dot{x}^\nu(0) \overset{(5.65)}{=} 0. \tag{5.40}$$

Therefore the acceleration vanishes. This means that the velocities of the particles don't change. The solution of the equations of motion (5.36) thus reads

$$\frac{dx^i}{d\tau} = 0 \quad \Rightarrow \quad x^i(\tau) = \text{const.} \tag{5.41}$$

In the chosen coordinates the particles in the field of the gravitational wave can thus be described by constant spatial coordinates. However, this does not mean that the particles are at rest. In fact their distances vary due to the time dependence of the metric tensor $g_{\mu\nu}$ as in Eq. (5.35).

Consider now particles which are arranged on a circle in the x^1-x^2-plane. The particles are initially at rest on a circle $((x^1)^2 + (x^2)^2 = R^2)$. We want to examine the effect of an incident gravitational wave along the x^3-axis. To do so, we write (5.35) in the form

$$\begin{aligned} ds^2 &= c^2dt^2 - dl^2 - (dx^3)^2 \\ \text{with } dl^2 &= (\delta_{mn} - h_{mn}(t))\, dx^m dx^n \quad (m, n = 1, 2). \end{aligned} \tag{5.42}$$

In the x^1-x^2-plane we have

$$h_{mn}(t) = h_{mn}(x^3 = 0, t) = \begin{pmatrix} e_{11} & e_{12} \\ e_{12} & -e_{11} \end{pmatrix} \exp(-i\omega t) + \text{c.c.} \tag{5.43}$$

where $\omega^2 = c^2k^2$. With (5.42) we can compute the physical distance ρ of a particle p from the center of the circle. According to Eq. (5.41) the coordinates x^1_p and x^2_p of the particle are constant. We insert in (5.42) the finite values of the coordinates x^m_p of p instead of dx^m (this is allowed because the metric coefficients do not depend on x^1 and x^2):

$$\rho^2 = (\delta_{mn} - h_{mn}(t))\, x^m_p x^n_p \quad (m, n = 1, 2) \tag{5.44}$$

$$\text{with } x^1_p = R\cos\varphi, \quad x^2_p = R\sin\varphi. \tag{5.45}$$

Using Eqs. (5.43)-(5.45) we find the solution

$$\rho^2 = R^2 \begin{cases} 1 - 2h\cos(2\varphi)\cos(\omega t) & \text{if } e_{11} = h,\ e_{12} = 0, \\ 1 - 2h\sin(2\varphi)\cos(\omega t) & \text{if } e_{11} = 0,\ e_{12} = h. \end{cases} \tag{5.46}$$

The $\cos(\omega t)$ term comes from $e^{-i\omega t} + \text{c.c.}$, whereas the $\cos(2\varphi)$ term, for example, comes from $\cos^2\varphi - \sin^2\varphi = \cos(2\varphi)$. The distinction that we made in (5.46) concerns the two possible linear polarization states.

Unlike to the coordinates x^1, x^2 which are constant, the physical variables $x = \rho\cos\varphi$ and $y = \rho\sin\varphi$ describe the distance relative to the center. The physical oscillations lead to a particle configuration which is an ellipse with very tiny eccentricities ($h \ll 1$). From the type of oscillation one can infer the polarization of the incoming wave. The two independent polarization states form an angle of $\frac{\pi}{4}$. Therefore the oscillations correspond to a quadrupole moment of the mass distribution: the gravitational waves induce a quadrupole oscillation of the mass distribution. Conversely we expect that mass distributions with oscillating quadrupole moment should emit gravitational waves. In order to study this phenomenon further, we have to learn more about the energy and momentum of a gravitational wave.

5.4 Energy and Momentum of a Gravitational Wave

We now want to determine the energy-momentum tensor of a gravitational wave, which is a solution of the free field equation up tofirst order in h ($|h_{\mu\nu}| \ll 1$)

$$R^{(1)}_{\mu\nu} = 0. \tag{5.47}$$

The solution to this equation is the wave solution that we derived before:

$$h_{\mu\nu} = e_{\mu\nu} \exp\left(-ik_\lambda x^\lambda\right) + \text{c.c.} \tag{5.48}$$

The energy-momentum tensor of a gravitational field is known from Eq. (3.12):

$$t^{\text{grav.}}_{\mu\nu} = \frac{c^4}{8\pi G}\left(R^{(2)}_{\mu\nu} - \frac{(g_{\mu\nu}R)^{(2)}}{2}\right) \tag{5.49}$$

with $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Using $R^{(0)}_{\mu\nu} = R^{(1)}_{\mu\nu} = 0$ and $R = g^{\rho\sigma}R_{\rho\sigma}$ we get

$$\begin{aligned} t^{\text{grav.}}_{\mu\nu} &= \frac{c^4}{16\pi G}\left[2R^{(2)}_{\mu\nu} - \eta_{\mu\nu}\eta^{\rho\sigma}R^{(2)}_{\sigma\rho} + \eta_{\mu\nu}h^{\rho\sigma}R^{(1)}_{\sigma\rho} - h_{\mu\nu}\eta^{\rho\sigma}R^{(1)}_{\sigma\rho}\right] \\ &= \frac{c^4}{16\pi G}\left[2R^{(2)}_{\mu\nu} - \eta_{\mu\nu}\eta^{\rho\sigma}R^{(2)}_{\sigma\rho}\right] \end{aligned} \tag{5.50}$$

where we used $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ which follows from $g^{\mu\nu}g_{\nu\lambda} = \delta^\mu_\lambda$. In order to consider this expression further, we need the Ricci tensor $R^{(2)}_{\mu\nu}$. In order to be able to calculate the Ricci tensor

$$R^{(2)}_{\mu\kappa} = \left(g^{\lambda\nu}R_{\lambda\mu\nu\kappa}\right)^{(2)} = \eta^{\lambda\nu}R^{(2)}_{\lambda\mu\nu\kappa} - h^{\lambda\nu}R^{(1)}_{\lambda\mu\nu\kappa} \tag{5.51}$$

we need the Riemann tensor which is given by:

$$\begin{aligned} R_{\lambda\mu\nu\kappa} = \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\mu \partial x^\kappa} + \frac{\partial^2 g_{\mu\kappa}}{\partial x^\lambda \partial x^\nu} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda \partial x^\kappa} - \frac{\partial^2 g_{\lambda\kappa}}{\partial x^\mu \partial x^\nu}\right) + \\ + g_{\eta\sigma}\left(\Gamma^\eta{}_{\nu\lambda}\Gamma^\sigma{}_{\mu\kappa} - \Gamma^\eta{}_{\kappa\lambda}\Gamma^\sigma{}_{\mu\nu}\right). \end{aligned} \tag{5.52}$$

The first line of this expression (which contains only first order terms in h) gives rise to the second term in (5.51) with $g_{ij} = h_{ij}$. The second line (which is of second order in h) gives rise to the first term in (5.51) if we insert the Christoffel symbols

$$\Gamma^{\sigma}{}^{(1)}_{\mu\nu} = \frac{1}{2}\left(\frac{\partial h^\sigma{}_\mu}{\partial x^\nu} + \frac{\partial h^\sigma{}_\nu}{\partial x^\mu} - \frac{\partial h_{\mu\nu}}{\partial x_\sigma}\right). \tag{5.53}$$

This yields the first term in (5.51) if we multiply with $\eta_{\mu\nu}$. In total, we get the following expression for the Ricci tensor that we searched for in order to evaluate Eq. (5.50):

$$\begin{aligned} R^{(2)}_{\mu\kappa} = &-\frac{h^{\lambda\nu}}{2}\left[\frac{\partial^2 h_{\lambda\nu}}{\partial x^\mu \partial x^\kappa}+\frac{\partial^2 h_{\mu\kappa}}{\partial x^\lambda \partial x^\nu}-\frac{\partial^2 h_{\mu\nu}}{\partial x^\lambda \partial x^\kappa}-\frac{\partial^2 h_{\lambda\kappa}}{\partial x^\mu \partial x^\nu}\right] \\ &+\frac{1}{4}\left[\frac{\partial h^\nu{}_\sigma}{\partial x^\nu}+\frac{\partial h^\nu{}_\sigma}{\partial x^\nu}-\frac{\partial h^\nu{}_\nu}{\partial x^\sigma}\right]\left[\frac{\partial h^\sigma{}_\mu}{\partial x^\kappa}+\frac{\partial h^\sigma{}_\kappa}{\partial x^\mu}-\frac{\partial h_{\mu\kappa}}{\partial x_\sigma}\right] \\ &-\frac{1}{4}\left[\frac{\partial h_{\sigma\kappa}}{\partial x^\lambda}+\frac{\partial h_{\sigma\lambda}}{\partial x^\kappa}-\frac{\partial h_{\lambda\kappa}}{\partial x^\sigma}\right]\left[\frac{\partial h^\sigma{}_\mu}{\partial x_\lambda}+\frac{\partial h^{\sigma\lambda}}{\partial x^\mu}-\frac{\partial h^\lambda{}_\mu}{\partial x_\sigma}\right]. \end{aligned} \tag{5.54}$$

The first term in the second line of this expression vanishes because of the gauge condition (3.18). The remaining terms are quadratic in h and of the form

$$\begin{aligned} &\left[e_{\mu\nu}\exp(-ik_\lambda x^\lambda)+\text{c.c.}\right]\left[e_{\sigma\kappa}\exp(-ik_\lambda x^\lambda)+\text{c.c.}\right] \\ &= e_{\mu\nu}e_{\sigma\kappa}\left[\exp(2ix_\lambda k^\lambda)+2+\exp(-2ix_\lambda k^\lambda)\right]. \end{aligned} \tag{5.55}$$

We can see that there appear on the one hand oscillating terms of the form $\exp(\pm 2ik_\lambda x^\lambda)$ and on the other hand there are also terms which do not depend on the coordinates x^μ at all. If we average over time, the oscillating terms drop out (their average over time is zero) such that we are left with terms of the form

$$\left\langle \left[e_{\mu\nu}\exp(-ik_\lambda x^\lambda)+\text{c.c.}\right]\left[e_{\sigma\kappa}\exp(-ik_\lambda x^\lambda)+\text{c.c.}\right]\right\rangle = 2\Re\left(e^*_{\mu\nu}e_{\sigma\kappa}\right)\ , \tag{5.56}$$

where $\langle\cdot\rangle$ denotes time-average and $\Re$ is the real part. For plane wave solutions, derivatives correspond to multiplication with k, so

$$\frac{\partial h_{\mu\nu}}{\partial x^\lambda} = -ik_\lambda h_{\mu\nu}. \tag{5.57}$$

We can now plug all these terms which are quadratic in h into Eq. (5.54), replacing all partial derivatives by factors of k:

$$\begin{aligned} \langle R^{(2)}_{\mu\kappa}\rangle = \Re\Big[&(e^{\lambda\nu})^*\left(k_\mu k_\kappa e_{\lambda\nu}+k_\lambda k_\nu e_{\mu\kappa}-k_\lambda k_\kappa e_{\mu\nu}-k_\mu k_\nu e_{\lambda\kappa}\right) \\ &+\left(e^\lambda{}_\rho k_\lambda-\frac{1}{2}e_\lambda{}^\lambda k_\rho\right)^*\left(k_\mu e^\rho{}_\kappa+k_\kappa e^\rho{}_\mu-k^\rho e_{\mu\kappa}\right) \\ &-\frac{1}{2}\left(k_\lambda e_{\sigma\kappa}+k_\kappa e_{\sigma\lambda}-k_\sigma e_{\kappa\lambda}\right)^*\cdot\left(k^\lambda e^\sigma{}_\mu+k_\mu e^{\sigma\lambda}-k^\sigma e^\lambda{}_\mu\right)\Big]. \end{aligned} \tag{5.58}$$

Using the relation: $2\partial_\mu h^\mu{}_\nu = \partial_\nu h^\mu{}_\mu$ (which follows from the gauge condition (3.18)) one can simplify the above expressions. For instance:

$$(e^{\lambda\mu})^* k_\kappa k_\lambda e_{\mu\nu} = \frac{1}{2}(e^\lambda{}_\lambda)^* k^\mu k_\kappa e_{\mu\nu} = \frac{1}{4}k_\kappa k_\nu\left|e^\lambda{}_\lambda\right|^2. \tag{5.59}$$

Imposing the null condition $k_\mu k^\mu = 0$ we obtain

$$\langle R^{(2)}_{\mu\kappa}\rangle = \frac{1}{2}k_\mu k_\kappa\left[(e^{\lambda\nu})^* e_{\lambda\nu}-\frac{1}{2}|e^\lambda{}_\lambda|^2\right]. \tag{5.60}$$

Thus the energy-momentum tensor (5.49) or (5.50) of the gravitational wave reads

$$\boxed{t_{\mu\nu}^{\text{grav.}} = \frac{c^4}{16\pi G} k_\mu k_\nu \left[(e^{\lambda\kappa})^* e_{\lambda\kappa} - \frac{1}{2} |e^\lambda{}_\lambda|^2 \right]} \tag{5.61}$$

where we used that

$$\eta_{\mu\nu}\eta^{\rho\sigma}\langle R^{(2)}_{\rho\sigma}\rangle \propto \eta_{\mu\nu}\eta^{\rho\sigma} k_\rho k_\sigma = \eta_{\mu\nu} k^\sigma k_\sigma = 0. \tag{5.62}$$

We can further simplify the energy-momentum tensor by specializing to the case of linearly polarized waves with either $e_{11} = -e_{22} = h$, $e_{12} = e_{21} = 0$ or $e_{11} = -e_{22} = 0$, $e_{12} = e_{21} = h$:

$$\boxed{t_{\mu\nu}^{\text{grav.}} = \frac{c^4}{8\pi G} k_\mu k_\nu h^2.} \tag{5.63}$$

Energy in this formula, being proportional to frequency squared is exactly the type of relation that we would intuitively expect. Furthermore it is clear that $t_{\mu\nu} \propto k_\mu k_\nu$ because t_{0i} is the current of momentum which should be proportional to k_i. We see immediately that measuring such energies will be extremely difficult because h^2 is very small. A wave propagating in the x^3-direction has the wavevector $k_\mu = \left(\frac{\omega}{c}, 0, 0, \frac{\omega}{c}\right)$. The energy current density

$$\Phi_{\text{grav.}} = c t^{03}_{\text{grav.}} = \frac{c^5}{8\pi G} k^0 k^3 h^2 \tag{5.64}$$

for such a wave is then given by

$$\Phi_{\text{grav.}} = \frac{\text{energy}}{\text{time} \cdot \text{surface}} = \frac{c^3}{8\pi G}\omega^2 h^2. \tag{5.65}$$

5.5 Quadrupole Radiation

Oscillating charge distributions emit electromagnetic waves. In analogy we expect oscillating mass distributions to emit gravitational waves. We quickly repeat the case of electromagnetic dipole radiation before turning to the case of oscillating mass distributions.

5.5.1 Dipole Radiation in Electromagnetism

In electromagnetism one finds that an oscillating dipole moment

$$\boldsymbol{p}(t) = \boldsymbol{p_0} \exp(-i\omega t) + \text{c.c.} \tag{5.66}$$

emits electromagnetic waves whose power P per solid angle is given by

$$\frac{dP}{d\Omega} = \frac{\omega^4}{8\pi c^3} |\boldsymbol{p}|^2 \sin^2\theta \tag{5.67}$$

where θ is the angle between $\boldsymbol{p}$ and $\boldsymbol{k}$ where $\boldsymbol{k}$ is the direction of propagation. This is sketched in Fig. 3. The total emitted power can be obtained by integrating in θ:

$$P = \frac{\omega^4}{3c^3} |\boldsymbol{p}|^2. \tag{5.68}$$

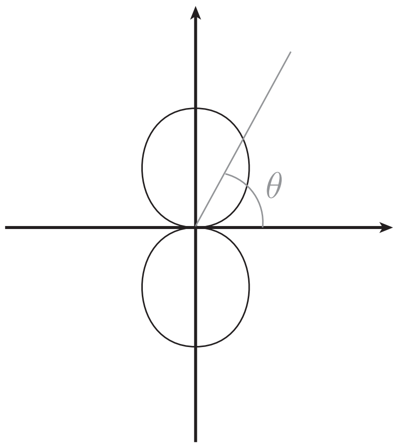

Figure 3: Sketch of the waves that are emitted by a dipole in electromagnetism.

5.5.2 Gravitational Quadrupole Radiation

The computation of the emitted gravitational radiation is similar to electromagnetism but also more involved since the source terms are rank 2 tensors. We will proceed with the following steps:

1. Calculate the asymptotic fields emitted by a source $T_{\mu\nu}$.
2. Reduce the result to spatial components.
3. Apply the long wavelength approximation.

The setup is sketched in Fig. 4. In contrast to the electromagnetic case there is no gravitational dipole

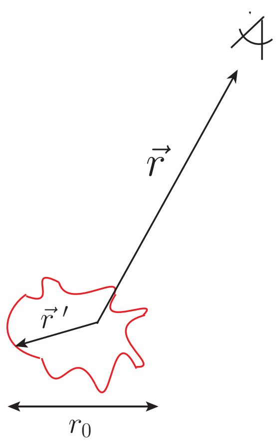

Figure 4: Sketch of the setup for gravitational wave emission. The source has spatial extent r_0. The observer is at position $\boldsymbol{r}$.

radiation. The density is given by

$$\rho(\boldsymbol{r},t)=\rho(\boldsymbol{r})\exp(-i\omega t)+\text{c.c.}\quad\Rightarrow\quad \boldsymbol{p}=\int d^3r\;\boldsymbol{r}\rho(\boldsymbol{r})=M\boldsymbol{R}_{\text{c.m.}} \tag{5.69}$$

where M is the total mass and $\boldsymbol{R}_{\text{c.m.}}$ is the center of mass. If we choose the center of mass system as the inertial system then $\boldsymbol{p}=0$. Consequently $\boldsymbol{p}=0$ in all inertial systems. This is not possible in electromagnetism. We shall now assume an oscillatory mass distribution of the form

$$T_{\mu\nu}(\boldsymbol{r},t)=T_{\mu\nu}(\boldsymbol{r})\exp(-i\omega t)+\text{c.c.}\quad\begin{cases}\neq 0 & \text{if } r\leq r_0\\ =0 & \text{otherwise.}\end{cases} \tag{5.70}$$

This is only a single Fourier component. Thus a generalization is possible by integrating over ω. According to (3.22) the retarded potentials are given by

$$h_{\mu\nu}(\boldsymbol{r},t)=-\frac{4G}{c^4}\exp(-i\omega t)\int d^3r'\;S_{\mu\nu}(\boldsymbol{r}')\frac{\exp(ik|\boldsymbol{r}-\boldsymbol{r}'|)}{|\boldsymbol{r}-\boldsymbol{r}'|}+\text{c.c.} \tag{5.71}$$

where we used

$$-i\omega t_{\text{r}}=-i\omega\left[t-\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right]=-i\omega t+ik|\boldsymbol{r}-\boldsymbol{r}'| \tag{5.72}$$

to obtain the phase factors. Furthermore we have introduced

$$S_{\mu\nu}(\boldsymbol{r})=T_{\mu\nu}(\boldsymbol{r})-\frac{1}{2}\eta_{\mu\nu}T(\boldsymbol{r}). \tag{5.73}$$

We now assume $r_0\ll r$ with $k=\frac{\omega}{c}=\frac{2\pi}{\lambda}$. For large distances we have $|\boldsymbol{r}|\gg r_0$ and thus

$$|\boldsymbol{r}-\boldsymbol{r}'|=r-\frac{\boldsymbol{r}\cdot\boldsymbol{r}'}{r}+\ldots=r\left[1+\mathcal{O}\left(\frac{r'}{r}\right)\right] \tag{5.74}$$

and

$$\exp\left[ik|\boldsymbol{r}-\boldsymbol{r}'|\right]=\exp[ikr]\exp[-i\boldsymbol{k}\boldsymbol{r}']\left[1+\mathcal{O}\left(\frac{r'}{r}\right)\right] \tag{5.75}$$

where we defined $\boldsymbol{k}=k\frac{\boldsymbol{r}}{r}=k\boldsymbol{e}_{\boldsymbol{r}}$. This way we obtain for (5.71)

$$h_{\mu\nu}(\boldsymbol{r},t)=-\frac{4G}{c^4}\frac{1}{r}\exp\left[-ik_\lambda x^\lambda\right]\underbrace{\int d^3r'\;S_{\mu\nu}(\boldsymbol{r}')\exp\left[-i\boldsymbol{k}\boldsymbol{r}'\right]}_{=:S_{\mu\nu}(\boldsymbol{k})}+\text{c.c.} \tag{5.76}$$

where $S_{\mu\nu}(\boldsymbol{k})$ is the spatial Fourier transform of $S_{\mu\nu}(\boldsymbol{r})$. This yields

$$h_{\mu\nu}(\boldsymbol{r},t)=e_{\mu\nu}(\boldsymbol{r},\omega)\exp\left[-ik_\lambda x^\lambda\right]+\text{c.c.}. \tag{5.77}$$

The amplitudes are defined as

$$e_{\mu\nu}(\boldsymbol{r},\omega)=-\frac{4G}{c^4}\frac{1}{r}S_{\mu\nu}(\boldsymbol{k})=-\frac{4G}{c^4}\frac{1}{r}\left[T_{\mu\nu}(\boldsymbol{k})-\frac{\eta_{\mu\nu}}{2}T(\boldsymbol{k})\right] \tag{5.78}$$

and are proportional to $\frac{1}{r}$. They depend on $\frac{\boldsymbol{r}}{r} = \boldsymbol{e_r}$ and ω via $\boldsymbol{k} = k\boldsymbol{e_r}$. The energy current passing through a surface element $r^2 d\Omega$ is given by

$$dP = ct_{0i}^{\text{grav.}} df^i = ct_{0i}^{\text{grav.}} \frac{x^i}{r} r^2 d\Omega. \tag{5.79}$$

Plugging in Eq. (5.61) for $t_{\mu\nu}^{\text{grav.}}$ we obtain

$$\frac{dP}{d\Omega} = c\frac{c^4}{16\pi G}\frac{k_0 k_i x^i}{r} r^2 \left[(e^{\lambda\nu})^* e_{\lambda\nu} - \frac{1}{2}|e^{\lambda}{}_{\lambda}|^2 \right]. \tag{5.80}$$

In the derivation of (5.61) we assumed $e_{\mu\nu} = \text{const.}$, whereas here we have $e_{\mu\nu} \propto \frac{1}{r}$. The energy-momengum tensor $t_{\mu\nu}^{\text{grav.}}$ contains partial derivatives of the $h_{\mu\nu}$ which would lead to additional terms $\propto \frac{1}{r}$. With $e_{\mu\nu} = \text{const.}$, the derivatives just lead to factors of $k_\mu \propto \frac{1}{\lambda}$. In the far field and distant observer approximation we have $r \gg \lambda$ and can thus neglect the additional terms since $\frac{1}{r} \ll \frac{1}{\lambda}$. Using

$$\frac{k_i x^i}{r} = \frac{\boldsymbol{k} \cdot \boldsymbol{r}}{r} = k = \frac{\omega}{c} \tag{5.81}$$

in Eq. (5.80) we get

$$\frac{dP}{d\Omega} = \frac{G\omega^2}{\pi c^5}\left[T^{\mu\nu}(\boldsymbol{k})^* T_{\mu\nu}(\boldsymbol{k}) - \frac{1}{2}|T(\boldsymbol{k})|^2 \right] \tag{5.82}$$

where $T_{\mu\nu}(\boldsymbol{k})$ is the Fourier transform of the source distribution.

We proceed with the second step as outlined in the beginning: we want to reduce our results to spatial components. The source distribution can be written as

$$T_{\mu\nu}(\boldsymbol{r}, t) = \frac{1}{(2\pi)^3}\int d^3k \, T^{\mu\nu}(\boldsymbol{k}) \exp\left[-ik_\lambda x^\lambda\right] + \text{c.c.}. \tag{5.83}$$

For weak fields the covariant derivative in the energy-momentum conservation simplifies to an ordinary derivative and the continuity equation reads

$$k_\mu T^{\mu\nu}(\boldsymbol{k}) = 0. \tag{5.84}$$

In particular we find for $\nu = 0$ and $\nu = i$

$$k_0 T^{00} = -k_j T^{0j} \quad \text{and} \quad k_0 T^{i0} = -k_j T^{ij}. \tag{5.85}$$

We can define a three dimensional unit vector $\hat{k}_i = \frac{k_i}{k}$ and obtain for (5.85)

$$T^{0i} = T^{i0} = -\hat{k}_j T^{ij}, \tag{5.86}$$

$$T^{00} = \hat{k}_i \hat{k}_j T^{ij}. \tag{5.87}$$

All non-spatial components in (5.82) can thus be eliminated and we calculate

$$\begin{aligned} T^{\mu\nu *}T_{\mu\nu} &= \eta_{\mu\rho}\eta_{\nu\sigma}T^{\mu\nu *}T^{\rho\sigma} \\ &= T^{00*}T^{00} - 2\sum_i T^{0i*}T^{0i} + \sum_{i,j} T^{ij*}T^{ij} \\ &= \hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m T^{ij*}T^{lm} - 2\hat{k}_j\hat{k}_m\delta_{il}T^{ij*}T^{lm} + \delta_{il}\delta_{jm}T^{ij*}T^{lm}, \end{aligned} \tag{5.88}$$

$$\begin{aligned} T^{\lambda}{}_{\lambda} &= \eta_{\lambda\rho}T^{\rho\lambda} \\ &= T^{00} - \sum_i T^{ii} \\ &= \hat{k}_i\hat{k}_j T^{ij} - \delta_{ij}T^{ij}, \end{aligned} \tag{5.89}$$

$$|T^{\lambda}{}_{\lambda}|^2 = \hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m T^{ij*}T^{lm} - \delta_{ij}\hat{k}_l\hat{k}_m T^{ij*}T^{lm} - \delta_{lm}\hat{k}_i\hat{k}_j T^{ij*}T^{lm}. \tag{5.90}$$

Inserting these expressions into (5.82) we get

$$\frac{dP}{d\Omega} = \frac{G\omega^2}{\pi c^5}\Lambda_{ij,lm}T^{ij}(\boldsymbol{k})^*\, T^{lm}(\boldsymbol{k}) \tag{5.91}$$

where we introduced

$$\Lambda_{ij,lm}(\theta,\varphi) = \delta_{il}\delta_{jm} - \frac{1}{2}\delta_{ij}\delta_{lm} - 2\delta_{il}\hat{k}_j\hat{k}_m + \frac{1}{2}\delta_{ij}\hat{k}_l\hat{k}_m + \frac{1}{2}\delta_{lm}\hat{k}_i\hat{k}_j + \frac{1}{2}\hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m. \tag{5.92}$$

Having reduced the formula for the radiated power to spatial components, we now turn to the last step that we outlined in the beginning: we apply the long wavelength approximation, i.e. we assume $\lambda \gg r_0$ which simplifies the energy-momentum tensor as follows:

$$\begin{aligned} T^{ij}(\boldsymbol{k}) &= \int d^3r\, T^{ij}(\boldsymbol{r})\exp(-i\boldsymbol{k}\boldsymbol{r}) \\ &= \int d^3r\, T^{ij}(\boldsymbol{r})(1 - i\boldsymbol{k}\boldsymbol{r} + ...) \\ &\approx \int d^3r\, T^{ij}(\boldsymbol{r}) \qquad =: -\frac{\omega^2}{2}Q^{ij}. \end{aligned} \tag{5.93}$$

The object Q^{ij} will turn out to be a quadrupole tensor. From covariant conservation of energy-momentum, $T^{\mu\nu}{}_{,\nu} = 0$, we get

$$\partial_j T^{ij}(\boldsymbol{r},t) = -\partial_0 T^{i0}(\boldsymbol{r},t) \quad \text{and} \quad \partial_i T^{0i}(\boldsymbol{r},t) = -\partial_0 T^{00}(\boldsymbol{r},t). \tag{5.94}$$

Using Eq. (5.70) we obtain

$$\partial_i\partial_j T^{ij}(\boldsymbol{r},t) = \partial_0^2 T^{00}(\boldsymbol{r},t) = -\frac{\omega^2}{c^2}T^{00}(\boldsymbol{r},t) \tag{5.95}$$

$$\Rightarrow \qquad \partial_i\partial_j T^{ij}(\boldsymbol{r}) = -\frac{\omega^2}{c^2}T^{00}(\boldsymbol{r}). \tag{5.96}$$

Since we are in the non-relativistic regime ($\lambda \gg r_0$, $v \ll c$) we have $T^{00} \simeq \rho c^2$. Therefore Eq. (5.93) yields

$$2 \int d^3r\, T^{ij}(\boldsymbol{r}) = \int d^3r\, x^i x^j \left(\partial_k \partial_l T^{lk}(\boldsymbol{r})\right) = -\frac{\omega^2}{c^2} \int d^3r\, x^i x^j T^{00}(\boldsymbol{r}) \tag{5.97}$$

where we integrated by parts twice in the first step and used Eq. (5.96) in the second step. According to the definition in Eq. (5.93), we find

$$\boxed{Q^{ij} = \int d^3r\, x^i x^j \rho(\boldsymbol{r}) = \frac{1}{c^2} \int d^3r\, x^i x^j T^{00}(\boldsymbol{r})} \tag{5.98}$$

which we can obviously interpret as the quadrupole tensor of the mass distribution[10]. Because we are in almost Minkowskian spacetime, we can compute Q^{ij} in three-dimensional Cartesian coordinate. Inserting (5.93) into (5.91) we get

$$\boxed{\frac{dP}{d\Omega} = \frac{G\omega^6}{4\pi c^5} \Lambda_{ij,lm} Q^{ij*} Q^{lm}}. \tag{5.99}$$

We observe that the corresponding formula in electrodynamics looks very similar but it depends on ω^4 rather than ω^6. This is just a reflection of the fact that the electromagnetic radiation is dipole radiation whereas gravitational dipole radiation does not exist (the dipole moment of any mass distribution vanishes in the center of mass system).

Furthermore we note that Q^{ij} are constants (they do not depend on θ or φ). The complete angular dependence is encoded in $\Lambda_{ij,lm}$ in which the vector $\hat{k}$ appears. This is the unit vector which indicates the direction from the mass distribution to the observer and therefore clearly depends on θ and φ:

$$(\hat{k}^i) = (\hat{k}_x, \hat{k}_y, \hat{k}_z) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta). \tag{5.100}$$

This simplifies calculations, of course, because the quadrupole moment can be calculated once and forever and the specific angular dependence is only to be considered in the form of $\Lambda_{ij,lm}$.

As an example we consider a quadrupole mass distribution in the principal axis system: the only non-vanishing elements are the diagonal, $Q_{ij} = \mathrm{diag}(Q_{11}, Q_{22}, Q_{33})$. In this case the only non-vanishing terms in (5.99) are those with $i = j$ and $l = m$. This means that there appear only even powers of $\hat{k}_x$, $\hat{k}_y$, $\hat{k}_z$ in Eq. (5.92). The emitted power has an angular dependence of the form

$$\frac{dP}{d\Omega} \sim a_1 \cos^4\theta + a_2 \cos^2\theta \sin^2\theta + a_3 \sin^4\theta. \tag{5.101}$$

To get the total emitted power, we have to integrate Eq. (5.99) over $d\Omega$:

$$\int d\Omega\, \Lambda_{ij,lm} = \frac{2\pi}{15}\left(11\delta_{il}\delta_{jm} - 4\delta_{ij}\delta_{lm} + \delta_{im}\delta_{jl}\right) \tag{5.102}$$

[10] In the literature one also finds different definitions of this tensor. For example, one can define a traceless version where $x^i x^j$ is replaced by $x^i x^j - \frac{1}{3} r^2 \delta^{ij}$.

where we used

$$\int d\Omega\ \hat{k}_i\hat{k}_j = \frac{4\pi}{3}\delta_{ij}, \tag{5.103}$$

$$\int d\Omega\ \hat{k}_i\hat{k}_j\hat{k}_l\hat{k}_m = \frac{4\pi}{15}\left(\delta_{ij}\delta_{lm} + \delta_{il}\delta_{jm} + \delta_{im}\delta_{jl}\right). \tag{5.104}$$

Inserting this result into (5.99) we obtain for the total emitted power

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{2G\omega^6}{5c^5}\left(\sum_{i,j=1}^{3}|Q^{ij}|^2 - \frac{1}{3}\left|\sum_{i=1}^{3}Q^{ii}\right|^2\right). \tag{5.105}$$

Note that Q_{ij} can be defined traceless such that the second sum in the brackets vanishes. Furthermore one can assume a more general time dependence than just $e^{-i\omega t}$. If we had defined the quadrupole moments in a traceless form,

$$Q^{ij}(t) = \int d^3x\ \left(x^i x^j - \frac{1}{3}r^2\delta^{ij}\right)\rho(t, \boldsymbol{x}), \tag{5.106}$$

then instead of Eq. (5.105) we would have found

$$\boxed{P = \frac{G}{5c^5}\langle\dddot{Q}^{kl}\dddot{Q}_{kl}\rangle} \tag{5.107}$$

where $\langle\cdot\rangle$ denotes a time average, for instance over one orbital period.[11] The third time derivatives $\dddot{Q}_{ij}$ in the above equation can be easily evaluated for a plane wave $Q_{ij} \propto \exp\left[-i\omega t\right]$ and yield $\dddot{Q}_{ij} \propto \omega^3$.

[11]In the literature, one finds also the formula $P = \frac{G}{45c^5}\langle\dddot{Q}^{kl}\dddot{Q}_{kl}\rangle$. The different prefactor arises if one defines the quadrupole tensor with an additional factor of 3 in the second term of Eq. (5.106).

6 Sources of Gravitational Waves

We want to consider different physical systems whose dynamics leads to the emission of gravitational waves.

6.1 Rigid Rotator

As a first example we consider the emission of gravitational waves by a rigid rotating body. Consider a coordinate system KS' with coordinates x'_m in which the body is fixed. In KS' the mass density $\rho'(\boldsymbol{r}')$ is time independent. We choose KS' such that the quadrupole tensor Θ'_{ij} is diagonal:

$$\Theta'_{ij} = \int d^3r'\ x'_i x'_j \rho'(\boldsymbol{r}') = \begin{pmatrix} I_1 & 0 & \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}. \tag{6.1}$$

We assume that the body rotates with angular velocity Ω around the x'_3-axis. The orthogonal transformation to an inertial system IS with coordinates x_n can be written as

$$x_n = \alpha^m{}_n(t) x'_m \qquad \text{with } \alpha^m{}_n(t) = \begin{pmatrix} \cos\Omega t & -\sin\Omega t & 0 \\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.2}$$

The tensor Θ_{ij} in IS reads

$$\begin{aligned} \Theta_{ij}(t) &= \int d^3r\ x_i x_j \rho(\boldsymbol{r}, t) \\ &= \int d^3r'\ (\alpha^n{}_i x'_n)\,(\alpha^m{}_j x'_m)\,\rho'(\boldsymbol{r}') \\ &= (\alpha(t)\Theta' \alpha^{\mathrm{T}}(t))_{ij} \end{aligned} \tag{6.3}$$

where we used $d^3r = d^3r'$ and $\rho'(\boldsymbol{r}') = \rho(\boldsymbol{r})$ since the density transforms as a scalar quantity. With Eqs. (6.1) and (6.2) we can compute (6.3):

$$\begin{aligned} \Theta_{11}(t) &= \frac{I_1 + I_2}{2} + \frac{I_1 - I_2}{2}\cos(2\Omega t) \\ \Theta_{12}(t) &= \frac{I_1 - I_2}{2}\sin(2\Omega t) \\ \Theta_{22}(t) &= \frac{I_1 + I_2}{2} - \frac{I_1 - I_2}{2}\cos(2\Omega t) \\ \Theta_{33}(t) &= I_3, \qquad \Theta_{13}(t) = \Theta_{31}(t) = 0. \end{aligned} \tag{6.4}$$

This is of the form

$$\Theta_{ij} = \text{const.} + [Q_{ij}\exp(-2i\Omega t) + \text{c.c.}] \tag{6.5}$$

$$\text{with } Q_{ij} = \frac{I_1 - I_2}{4} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.6}$$

Comparing Eqs. (6.3) and (6.5) we conclude that a rotating rigid body can be interpreted as a mass distribution whose rotational frequency Ω leads to gravitational waves of frequency $\omega = 2\Omega$. We introduce the moment of inertia I with respect to the rotation axis and the **ellipticity** of the body ε,

$$I = I_1 + I_2, \qquad \varepsilon = \frac{I_1 - I_2}{I_1 + I_2}. \tag{6.7}$$

We can then write

$$P = \frac{32 G \Omega^6}{5 c^5} \varepsilon^2 I^2 \tag{6.8}$$

which is clearly the type of formula that one expects for quadrupole radiation ($Q \sim \epsilon I$).

6.1.1 Example: The Double Star System

As an example we consider a binary star system (masses M_1 and M_2) rotating on a Keplerian ellipse. Assuming a circle with constant radius r, we can consider the system as a rigid rotator:

$$I \simeq I_1 = \frac{M_1 M_2}{M_1 + M_2} r^2, \qquad I_2 \simeq 0, \qquad \varepsilon \simeq 1. \tag{6.9}$$

The circular orbit is characterized by

$$\underbrace{\frac{M_1 M_2}{M_1 + M_2}}_{=\mu} \Omega^2 r = \frac{G M_1 M_2}{r^2} \qquad \Rightarrow \qquad \Omega^2 = \frac{GM}{r^3} \tag{6.10}$$

where μ is the reduced mass and $M = M_1 + M_2$. Inserting this into Eq. (6.8) we find

$$P = \frac{32 G^4 M_1^2 M_2^2 (M_1 + M_2)}{5 c^5 r^5} = \frac{32}{5} \frac{G^4}{c^5 r^5} M^3 \mu^2. \tag{6.11}$$

It is convenient to express the emitted power in terms of

$$\begin{aligned} L_\odot = \frac{c^5}{G} &= 3.63 \times 10^{59} \text{erg s}^{-1} \\ &= 2.03 \times 10^5 M_\odot c^2 \text{ s}^{-1} \end{aligned} \tag{6.12}$$

which yields

$$P = \frac{32}{5} \left(\frac{GM}{c^2 r} \right)^5 \frac{\mu^2}{M^2} L_\odot. \tag{6.13}$$

For example, coalescing neutron stars in the final stage have $r \approx R_S$ and thus $\frac{GM}{c^2 r} \sim \mathcal{O}(1)$ and $P \approx L_\odot$. In general, for order of magnitude estimates one can use

$$P = L_\odot \left(\frac{GM}{c^2 R} \right)^5 \tag{6.14}$$

for systems of typical scale R.

Due to the emission of gravitational waves the system loses energy and thus its distance R shrinks, until the two bodies coalesce after a time $t_{\text{spir.}}$ (**inspiral time**). In the Kepler problem the total energy is

$$E = -\frac{GM_1M_2}{2r}. \tag{6.15}$$

During the inspiral process, the system loses potential energy which is emitted in the form of gravitational waves. Thus $dE = -Pdt$ and

$$P = -\frac{dE}{dt} = -\frac{GM_1M_2}{2r^2}\frac{dr}{dt} = \frac{32}{5}\frac{G^4}{c^5}\frac{M_1^2M_2^2(M_1+M_2)}{r^5}. \tag{6.16}$$

Upon substituting $x(t) = [r(t)/r(0)]^4$ we can rewrite the last equality in (6.16) as

$$\frac{dx}{dt} = -\frac{256G^3}{5c^5}\frac{M_1M_2(M_1+M_2)}{r^4(0)} \equiv -\frac{1}{t_{\text{spir.}}}. \tag{6.17}$$

This is solved by $x = 1 - t/t_{\text{spir.}}$ such that we find

$$\boxed{r(t) = r(0)\left(1 - \frac{t}{t_{\text{spir.}}}\right)^{1/4}.} \tag{6.18}$$

Next, we want to calculate the **strain** of such a system. We evaluate the expression (5.78) for $e_{\mu\nu}$ in analogy to what we did with Eq. (5.79). We can express $T_{\mu\nu}$ in terms of its spatial components ($k_0T^{0i} = k_jT^{ij}$ and $k_0T^{00} = k_iT^{0i}$) which in turn can be expressed in terms of Q_{ij}:

$$T_{ij}(\boldsymbol{k}) = -\frac{\omega^2}{2}Q_{ij}. \tag{6.19}$$

With the definitions in (6.6) and (6.8) we obtain

$$e_{11} = \frac{1}{2}\frac{GI\varepsilon}{c^4}(2\Omega)^2\frac{1}{D} \tag{6.20}$$

where D is the distance between source and observer.

If there are two polarizations and $e_{11} = e_{12}$ then we have for the dimensionless strain

$$h = \sqrt{e_{11}^2 + e_{12}^2} = 2\sqrt{2}\frac{GI\varepsilon}{c^4}\Omega^2\frac{1}{D}. \tag{6.21}$$

For a binary system characterized by masses $\tilde{M} = M_1 = M_2$ on an orbit of radius r, this yields

$$r^2\Omega^2 = \frac{G\tilde{M}}{r}, \quad I = \frac{\tilde{M}r^2}{2} \qquad \Rightarrow \qquad h \sim \frac{R_S^2}{Dr} \tag{6.22}$$

where $R_S = 2G\tilde{M}/c^2$ is the Schwarzschild radius of the system.

For two neutron stars with $\tilde{M} = 1.4M_\odot$, $r = 100$ km (i.e. $T \approx 10^{-2}$ s, $\Omega \approx 6 \times 10^2$ Hz) at a distance $D = 30000$ ly ≈ 10 kpc we get

$$h \approx 10^{-18} \quad \text{and} \quad P \approx 10^{52}\,\text{erg}\,\text{s}^{-1}. \tag{6.23}$$

This strain is the relative amplitude of the oscillation of a ruler's length when the gravitational wave passes through Earth.

6.2 The PSR 1913+16 System

The Hulse-Taylor binary system is a famous example of the confirmation of the gravitational wave theory. Russell Alan Hulse and Joseph Taylor were awarded the 1993 Nobel Prize for the discovery and measurement of this system. The system consists of two neutron stars, one of which is a pulsar whose radio signals have been observed for many years. From the phase shift of the pulsar one can infer the orbital parameters

$$\tau = 0.06 \text{ s},$$

$$T = 27906.980894 \pm 0.000002 \text{ s},$$

$$M_1 = (1.442 \pm 0.003) M_\odot,$$

$$M_2 = (1.386 \pm 0.003) M_\odot,$$

where T is the orbital time of the stars. Because of these very accurate pulse times, very precise astrophysical measurements can be done with this system. Let us ignore ellipticity of the orbit for an order of magnitude estimate. We obtain for the spiral time

$$t_{\text{spir.}} \sim 10^9 \text{ yr.} \tag{6.24}$$

From Kepler's law we have $T^2 \propto r^3$ such that

$$2\frac{dT}{T} = 3\frac{dr}{r}. \tag{6.25}$$

Furthermore Eq. (6.18) implies that

$$\left(\frac{dr}{dt}\right)_0 = -\frac{1}{4}\frac{r(0)}{t_{\text{spir.}}}. \tag{6.26}$$

Therefore

$$\frac{dT}{dt} = \frac{3}{2}\frac{dr}{dt}\frac{T}{r} = -\frac{3}{8}\frac{T}{t_{\text{spir.}}} \sim 10^{-12}. \tag{6.27}$$

Through the phase shift of the pulses one can measure the variation in the orbital time and finds experimentally

$$\frac{dT}{dt} = -(2.43 \pm 0.03) \times 10^{-12}. \tag{6.28}$$

This is the same order of magnitude as the above simplified GR estimate.

So far we approximated the actual orbit by a circular one. We want to refine the analysis by taking into account that the real orbit is actually elliptical in order to make more precise predictions. As we will see, this more detailed analysis yields very good agreement with the experimental data (6.28). Accounting for elliptical orbits (see the textbooks of Maggiore or Straumann for derivations), we have for the semi-major axis ($G = c = 1$)

$$a = -\frac{M_1 M_2}{2E} \tag{6.29}$$

where E is the total energy. Denoting by L the angular momentum, one finds for the squared eccentricity of the orbit

$$e^2 = 1 + \frac{2EL^2(M_1 + M_2)}{M_1^3 M_2^3}. \tag{6.30}$$

The orbit can be described as

$$r = \frac{a(1-e^2)}{1 + e\cos\vartheta}. \tag{6.31}$$

The emitted power reads

$$P = \frac{8M_1^2 M_2^2}{15a^2(1-e^2)^2} \left\langle \left[12(1+e\cos\vartheta)^2 + e^2\sin^2\vartheta\right] \dot{\vartheta} \right\rangle \tag{6.32}$$

where $\langle\cdot\rangle$ denotes the time average over a period T,

$$\langle\cdot\rangle = \frac{1}{T}\int_0^T (\cdot)dt = \frac{1}{T}\int_0^{2\pi} (\cdot)\frac{d\vartheta}{\dot{\vartheta}}. \tag{6.33}$$

Using Kepler's third law

$$T = \frac{2\pi a^{3/2}}{(M_1 + M_2)^{1/2}}, \tag{6.34}$$

after averaging one finds

$$P = \frac{32G^4 M_1^2 M_2^2 (M_1 + M_2)}{5c^5 a^5 (1-e^2)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right). \tag{6.35}$$

Then averaging

$$\frac{da}{dt} = \frac{M_1 M_2}{2E^2}\frac{dE}{dt} \tag{6.36}$$

over time, one finds

$$\left\langle \frac{da}{dt} \right\rangle = \frac{2a^2}{M_1 M_2}\left\langle \frac{dE}{dt} \right\rangle = -\frac{64G^4}{5c^5}\frac{M_1 M_2 (M_1 + M_2)}{a^3(1-e^2)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right). \tag{6.37}$$

Using Kepler's third law again, this yields

$$\left\langle \frac{\dot{T}}{T} \right\rangle = \frac{3}{2}\left\langle \frac{\dot{a}}{a} \right\rangle = -\frac{96}{5c^5}\frac{G^4}{a^4} M_1 M_2 (M_1 + M_2) f(e) \tag{6.38}$$

$$\text{where} \quad f(e) := \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right)\frac{1}{(1-e^2)^{7/2}}. \tag{6.39}$$

If we replace a by T in the right-hand side of Eq. (6.38), we conclude

$$\left\langle \frac{\dot{T}}{T} \right\rangle = -\frac{96}{5c^5}\frac{M_1 M_2}{(T/2\pi)^{8/5}(M_1 + M_2)^{1/3}} f(e). \tag{6.40}$$

Plugging in the measured ellipticity of $e \approx 0.617$, one gets

$$\dot{T}_{\text{theoretical}} = (-2.40243 \pm 0.00005) \times 10^{-12} \tag{6.41}$$

which is in perfect agreement with the measured data. This is indirect evidence for the existence of gravitational waves.

7 The post-Newtonian Approximation

Astrophysical sources of gravitational radiation are held together by gravitational forces. For a self-gravitating system of mass m we have

$$\left(\frac{v}{c}\right)^2 \sim \frac{R_\mathrm{S}}{d} \tag{7.1}$$

where R_S is the Schwarzschild radius and d is a typical size of the system. This relation follows immediately from $v^2 \approx Gm/r \approx Gm/d$. We note that R_S/d is a measure of the strength of the gravitational field close to the source. As soon as we consider v/c corrections to the orbital motion, for consistency we have to consider also $(R_\mathrm{S}/d)^{1/2}$ corrections to the metric (corrections to the flat background).

Moderately relativistic systems require a **post-Newtonian** treatment. The assumptions that we will use are

- The systems under consideration are moving slowly, weakly self-gravitating systems such that an expansion in v/c or $(R_\mathrm{S}/d)^{1/2}$ is possible.
- The energy-momentum tensor $T_{\mu\nu}$ has a spatially compact support ($T_{\mu\nu}(r) = 0$ for $r \geq d$).

If ω_S is a typical frequency of the system, then typical velocities are $v \approx \omega_\mathrm{S} d$. As we saw before, the frequency of the radiation of the emitted gravitational radiation is $\omega = 2\omega_\mathrm{S} \approx 2v/d$. In non-relativistic systems we have $v \ll c$, thus $c/v \gg 1$ and the wavelength of the emitted radiation satisfies $\lambda = c/\omega \sim cd/v \gg d$.

In analogy to the electromagnetic case, for non-relativisic sources it is convenient to distinguish between:

- *near field* regime ($r, d \ll \lambda$) where retardation is negligible and potentials are static,
- *far field* regime ($r \gg \lambda$) where retardation is crucial and we have waves.

The small parameter in powers of which we will perform an expansion, is[12]

$$\varepsilon \sim \left(\frac{R_\mathrm{S}}{d}\right)^{1/2} \sim \frac{v}{c}. \tag{7.2}$$

We demand that

$$\frac{|T^{ij}|}{T^{00}} \sim \mathcal{O}(\varepsilon^2). \tag{7.3}$$

For a fluid with pressure p and energy density ρ, we thus have

$$\frac{p}{\rho} \sim \varepsilon^2. \tag{7.4}$$

We expand the metric and the energy-momentum tensor in the near field regime in powers of ε.

As long as the emission of radiation is neglected, a classical system subject to conservative forces is invariant under time reversal. Under time reversal, g_{00} and g_{ij} are even (i.e. there appear even powers

[12] Note that some references also use the convention $\varepsilon \sim (v/c)^2$.

of v and thus of ε, as well) while g_{0i} is odd (i.e. only odd powers of v and thus of ε appear). By inspection of Einstein's field equations, one finds that in order to work consistently to a given order ε, if we expand g_{00} up to $\mathcal{O}(\varepsilon^n)$, then we have to expand g_{0i} up to $\mathcal{O}(\varepsilon^{n-1})$ and g_{ij} up to $\mathcal{O}(\varepsilon^{n-2})$.

The metric is expanded as follows:

$$\begin{aligned} g_{00} &= -1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + {}^{(6)}g_{00} + ... \\ g_{0i} &= \qquad\quad {}^{(3)}g_{0i} + {}^{(5)}g_{0i} + ... \\ g_{ij} &= \ \delta_{ij} + {}^{(2)}g_{ij} + {}^{(4)}g_{ij} + ... \end{aligned} \tag{7.5}$$

where ${}^{(n)}g_{\mu\nu}$ denotes terms of $\mathcal{O}(\varepsilon^n)$. Similarly, for the energy-momentum tensor:

$$\begin{aligned} T^{00} &= {}^{(0)}T^{00} + {}^{(2)}T^{00} + ... \\ T^{0i} &= \qquad {}^{(1)}T^{0i} + {}^{(3)}T^{0i} + ... \\ T^{ij} &= \qquad {}^{(2)}T^{ij} + {}^{(4)}T^{ij} + ... \end{aligned} \tag{7.6}$$

We now want to insert these expansions into Einstein's field equations and equate terms of the same order in ε. Considering $v \ll c$, the time derivatives of the metric are smaller than the spatial derivatives by $\mathcal{O}(\varepsilon)$:

$$\frac{\partial}{\partial t} = \mathcal{O}(v)\frac{\partial}{\partial x} \qquad \text{or} \qquad \partial_0 \sim \mathcal{O}(\varepsilon)\partial_i, \tag{7.7}$$

where we used that $\partial_0 = \frac{1}{c}\partial_t$. The d'Alembert operator applied to the metric, to lowest order becomes the Laplacian:

$$\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta\right) = \left(\mathcal{O}(\varepsilon^2) + 1\right)\Delta. \tag{7.8}$$

This means that retardation effects are small corrections.

Consequently, we also have to expand the geodesic equation

$$\frac{d^2x^i}{d\tau^2} = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{7.9}$$

In chapter 9.3 of GR I , we considered the Newtonian limit where we just had to use $g_{00} = -1 + {}^{(2)}g_{00}$, $g_{0i} = 0$ and $g_{ij} = \delta_{ij}$. It thus follows that the terms ${}^{(4)}g_{00}$, ${}^{(3)}g_{0i}$ and ${}^{(2)}g_{ij}$ give the first post-Newtonian order for which we use the notation 1PN. The terms ${}^{(6)}g_{00}$, ${}^{(5)}g_{0i}$, ${}^{(4)}g_{ij}$ give the 2PN approximation (which is already highly complicated), and so on.

7.1 The 1PN Approximation

It is useful to choose a simplifying gauge condition right from the beginning. A convenient choice is the **de Donder gauge condition**

$$\partial_\mu(\sqrt{-g}g^{\mu\nu}) = 0. \tag{7.10}$$

(this is a harmonic gauge condition, i.e. the coordinate functions satisfy the d'Alembert equation).

The next step is to insert Eqs. (7.5), (7.6) into Einstein's equations (together with (7.7) and (7.10)). We skip the explicit computations but state only the results[13]. One finds for ${}^{(2)}g_{00}$ the Newtonian equation

$$\Delta\left[{}^{(2)}g_{00}\right] = -\frac{8\pi G}{c^4}{}^{(0)}T^{00} \tag{7.11}$$

while the 1PN correction to the metric yields

$$\Delta\left[{}^{(2)}g_{ij}\right] = -\frac{8\pi G}{c^4}\delta_{ij}{}^{(0)}T^{00}, \tag{7.12}$$

$$\Delta\left[{}^{(3)}g_{0i}\right] = \frac{16\pi G}{c^4}{}^{(1)}T^{0i}, \tag{7.13}$$

$$\begin{aligned}\Delta\left[{}^{(4)}g_{00}\right] &= \partial_0^2\left[{}^{(2)}g_{00}\right] + {}^{(2)}g_{ij}\partial_i\partial_j\left[{}^{(2)}g_{00}\right] - \partial_i\left[{}^{(2)}g_{00}\right]\partial_i\left[{}^{(2)}g_{00}\right] \\ &\quad - \frac{8\pi G}{c^4}\left[{}^{(2)}T^{00} + {}^{(2)}T^{ii} - 2{}^{(2)}g_{00}{}^{(0)}T^{00}\right],\end{aligned} \tag{7.14}$$

where $\Delta = \delta^{ij}\partial_i\partial_j$ and the sum over repeated spatial indices is performed with δ^{ij}.

The solution of (7.11) with the boundary condition that the metric is asymptotically flat, is

$$ {}^{(2)}g_{00} = -2\phi \qquad \text{with } \phi(t,\boldsymbol{x}) = -\frac{G}{c^4}\int d^3x'\frac{{}^{(0)}T^{00}(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|} \tag{7.15}$$

where the Newtonian potential is $U = -c^2\phi$. Similarly Eqs. (7.12) and (7.13) are solved by

$$ {}^{(2)}g_{ij} = -2\phi\delta_{ij} \tag{7.16}$$

$$ {}^{(3)}g_{0i} = \xi_i \tag{7.17}$$

where we defined

$$\xi_i(\boldsymbol{x},t) = -\frac{4G}{c^4}\int d^3x'\frac{{}^{(1)}T^{0i}(\boldsymbol{x}',t)}{|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{7.18}$$

In order to solve (7.14), we replace ${}^{(2)}g_{00}$ on the right-hand side by -2ϕ and ${}^{2}g_{ij}$ by $-2\phi\delta_{ij}$. Furthermore we use the identity

$$(\boldsymbol{\nabla}\phi)^2 = \partial_i\phi\partial_i\phi = \frac{1}{2}\Delta(\phi^2) - \phi\Delta\phi. \tag{7.19}$$

and we introduce a new potential ψ such that

$$ {}^{(4)}g_{00} = -2(\phi^2+\psi). \tag{7.20}$$

Eq. (7.14) then reads

$$\Delta\psi = \partial_0^2\phi + \frac{4\pi G}{c^4}\left({}^{(2)}T^{00} + {}^{(2)}T^{ii}\right). \tag{7.21}$$

[13]For details, see section 9.1. in Weinberg's book or section 5.2 in Straumann's.

Using the boundary condition that ψ vanishes at spatial infinity, ψ can be written as

$$\psi(\boldsymbol{x},t) = -\int \frac{d^3x'}{|\boldsymbol{x}-\boldsymbol{x}'|}\left(\frac{1}{4\pi}\partial_0^2\phi + \frac{G}{c^4}\left[{}^{(2)}T^{00}(\boldsymbol{x}',t) + {}^{(2)}T^{ii}(\boldsymbol{x}',t)\right]\right). \tag{7.22}$$

Notice that ϕ and ξ_i are not independent due to the gauge condition (7.10) which imposes the contraint

$$4\partial_0\phi + \boldsymbol{\nabla}\cdot\boldsymbol{\xi} = 0. \tag{7.23}$$

From Eqs. (7.15) and (7.18) one can see that this condition is indeed satisfied due to energy-momentum conservation at the 1PN order (since $T^{\mu\nu}$ is covariantly conserved in the exact solution, it has to be conserved at all post-Newtonian orders independently).

We observe that ϕ, ψ, ξ_i are instantaneous potentials. Our order of approximation is thus insensitive to retardation effects.

Note also that g_{00} can be expressed very simply as

$$g_{00} = -e^{-2V/c^2} + \mathcal{O}(\varepsilon^6) \tag{7.24}$$

where $V = -c^2(\phi+\psi)$. This follows immediately if we expand the exponential and write (7.24) as

$$\begin{aligned} g_{00} &= -1 + \frac{2V}{c^2} - \frac{2V^2}{c^4} + \mathcal{O}\left(\frac{1}{c^6}\right) \\ &= -1 - 2(\phi+\psi) - 2(\phi+\psi)^2 + \mathcal{O}(\varepsilon^6) \\ &= -1 - 2\phi - 2(\phi^2+\psi) - 2(\psi^2 + 2\phi\psi) + \mathcal{O}(\varepsilon^6). \end{aligned} \tag{7.25}$$

Using that $\phi = \mathcal{O}(\varepsilon^2)$ and $\psi = \mathcal{O}(\varepsilon^4)$, we see that this is just

$$g_{00} = -1 + {}^{(2)}g_{00} + {}^{(4)}g_{00} + \mathcal{O}(\varepsilon^6). \tag{7.26}$$

Putting together (7.11) (with ${}^{(2)}g_{00} = -2\phi$) and (7.21), we have

$$\Delta(\phi+\psi) = \partial_0^2\phi + \frac{4\pi G}{c^4}\left[{}^{(0)}T^{00} + {}^{(2)}T^{00} + {}^{(2)}T^{ii}\right]. \tag{7.27}$$

To this order we can set $\partial_0^2\phi = \partial_0^2(\phi+\psi)$ and replace Δ by $\Box$. We then obtain

$$\Box V = -\frac{4\pi G}{c^2}\left[T^{00} + T^{ii}\right] \equiv -\frac{4\pi G}{c^2}\sigma \tag{7.28}$$

where we replaced ${}^{(0)}T^{00} + {}^{(2)}T^{00} \to T^{00}$ and ${}^{(2)}T^{ii} \to T^{ii}$. The solution of Eq. (7.28) is given by a retarded integral

$$V(\boldsymbol{x},t) = G\int d^3x'\,\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\underbrace{\left(T^{00}+T^{ii}\right)}_{=\sigma}\left(t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}, \boldsymbol{x}'\right) \tag{7.29}$$

and similarly for ξ_i defined as V_i, given by

$$V_i(\boldsymbol{x},t) = G\int d^3x'\,\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\sigma_i\left(t - \frac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}, \boldsymbol{x}'\right) \tag{7.30}$$

where $\sigma_i \equiv T^{0i}$.

To summarize, in harmonic coordinates the 1PN solution can be written in terms of two functions V and V_i in the following way:

$$\boxed{\begin{aligned} g_{00} &= -1 + \frac{2}{c^2}V - \frac{2}{c^4}V^2 + \mathcal{O}\left(\frac{1}{c^6}\right), \\ g_{0i} &= -\frac{4}{c^3}V_i + \mathcal{O}\left(\frac{1}{c^5}\right), \\ g_{ij} &= \delta_{ij}\left(1 + \frac{2}{c^2}V\right) + \mathcal{O}\left(\frac{1}{c^4}\right). \end{aligned}} \tag{7.31}$$

7.2 Motion of Test Particles in the 1PN Metric

To get the equations of motion of a particle of mass m in the near zone, we have to solve the geodesic equation which is encoded in the following action which reads in curved background

$$\begin{aligned} S &= -mc \int dt \left(-g_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\right)^{1/2} \\ &= -mc^2 \int dt \left(-g_{00} - 2g_{0i}\frac{v^i}{c} - g_{ij}\frac{v^i v^j}{c^2} + \dots\right)^{1/2}. \end{aligned} \tag{7.32}$$

We are interested in the equations of motion for a binary system. If we restrict ourselves to the lowest PN corrections, it is possible to treat the two masses as point-like.

In curved space the energy-momentum tensor of a set of point-like particles is given by

$$T^{\mu\nu} = \frac{1}{\sqrt{-g}} \sum_{a=1}^{N} \gamma_a m_a \frac{dx_a^\mu}{dt}\frac{dx_a^\nu}{dt}\delta^{(3)}\left(\boldsymbol{x} - \boldsymbol{x}_a(t)\right). \tag{7.33}$$

where the masses are denoted by m_a and x_a^μ are the coordinates ($a = 1, ..., N$). Furthermore we used the definitions

$$\gamma_a = \frac{1}{\sqrt{1 - \frac{v_a^2}{c^2}}} \qquad \text{and} \qquad p_a^\mu = \gamma_a m_a \frac{dx_a^\mu}{dt}. \tag{7.34}$$

In an N-body system ($a > 2$) the metric felt by a particle b is obtained by taking the energy-momentum tensor of all other particles as a source. This amounts to replacing $\sum_a$ by $\sum_{a(\neq b)}$ in Eq. (7.33). We expand the determinant of the metric to second order and using ${}^{(2)}g_{00} = -2\phi$, we get

$$-g = 1 - {}^{(2)}g_{00} + \sum_i {}^{(2)}g_{ii} = 1 - 4\phi. \tag{7.35}$$

Therefore the expansion of (7.33) gives

$$\boxed{\begin{aligned} {}^{(0)}T^{00}(\boldsymbol{x},t) &= \sum_{a(\neq b)} m_a c^2 \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(t)) \\ {}^{(2)}T^{00}(\boldsymbol{x},t) &= \sum_{a(\neq b)} m_a \left(\frac{1}{2}v_a^2+\phi c^2\right)\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(t)) \\ {}^{(1)}T^{0i}(\boldsymbol{x},t) &= c\sum_{a(\neq b)} m_a v_a^i \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(t)) \\ {}^{(2)}T^{ij}(\boldsymbol{x},t) &= \sum_{a(\neq b)} m_a v_a^i v_a^j \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}_a(t)). \end{aligned}} \tag{7.36}$$

Inserting these expressions into Eqs. (7.15), (7.18) and (7.22), one can obtain the metric in which the particle b propagates. Inserting this metric into (7.32), one can calculate its action S_b. The total action can then be obtained by summing over all particles, $S = \sum_b S_b$. Expanding the square root that appears in the integral of the action and demanding consistency of the expansion, only terms up to $\mathcal{O}\left((v/c)^4\right)$ give the 1PN correction.

In terms of the Lagrangian, one can verify the following results for the two-body system:

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{c^2}\mathcal{L}_1 \tag{7.37}$$

$$\text{with } \mathcal{L}_0 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{Gm_1 m_2}{r}, \tag{7.38}$$

$$\begin{aligned} \mathcal{L}_1 &= \frac{1}{8}m_1 v_1^4 + \frac{1}{8}m_2 v_2^4 + \\ &+ \frac{Gm_1 m_2}{2r}\left[3(v_1^2+v_2^2) - 7\boldsymbol{v}_1\boldsymbol{v}_2 - (\hat{\boldsymbol{r}}\boldsymbol{v}_1)(\hat{\boldsymbol{r}}\boldsymbol{v_2}) - \frac{G(m_1+m_2)}{r}\right]. \end{aligned} \tag{7.39}$$

where r denotes the separation vector between the two particles, $r = |\boldsymbol{r}|$ and $\hat{\boldsymbol{r}} = \frac{\boldsymbol{r}}{r}$. The Lagrangians $\mathcal{L}_0$ and $\mathcal{L}_1$ describe the Newtonian part and the first post-Newtonian correction, respectively.

The same can be obtained for N particles (**Einstein-Infeld-Hoffmann Lagrangian**) :

$$\mathcal{L}_0 = \sum_a \frac{1}{2}m_a v_a^2 + \sum_{a(\neq b)} \frac{Gm_a m_b}{2r_{ab}}, \tag{7.40}$$

$$\begin{aligned} \mathcal{L}_1 &= \sum_a \frac{1}{8}m_a v_a^4 - \sum_{a(\neq b)} \frac{Gm_a m_b}{4r_{ab}}\left[7\boldsymbol{v}_a\boldsymbol{v}_b + (\hat{\boldsymbol{r}}_{ab}\boldsymbol{v}_a)(\hat{\boldsymbol{r}}_{ab}\boldsymbol{v}_b)\right] + \\ &+ \frac{3G}{2}\sum_a\sum_{b\neq a}\frac{m_a m_b v_a^2}{r_{ab}} - \frac{G^2}{2}\sum_a\sum_{b\neq a}\sum_{c\neq a}\frac{m_a m_b m_c}{r_{ab}r_{ac}}, \end{aligned} \tag{7.41}$$

where $a = 1, ..., N$ labels the particles, r_{ab} is the distance between particles a and b, and $\hat{\boldsymbol{r}}_{ab}$ is the corresponding unit vector. From this Lagrangian one can derive the Euler-Lagrange equations of the N particle system including 1PN corrections. These equations are also called the **Einstein-Infeld-Hoffmann equations**. Denoting $r_{ab} = |\boldsymbol{x}_a - \boldsymbol{x}_b|$ and $\boldsymbol{x}_{ab} = \boldsymbol{x}_a - \boldsymbol{x}_b$, one finds after lengthy calculations

$$\dot{\boldsymbol{v}}_a = -G\sum_{b\neq a} m_b \frac{\boldsymbol{x}_{ab}}{r_{ab}^3}\Bigg[1 - 4G\sum_{c\neq a}\frac{m_c}{r_{ac}} + G\sum_{c\neq a,b} m_c\left(-\frac{1}{r_{bc}} + \frac{\boldsymbol{x}_{ab}\boldsymbol{x}_{bc}}{2r_{bc}^3}\right) -$$

$$- 5G\frac{m_a}{r_{ab}} + \boldsymbol{v}_a^2 - 4\boldsymbol{v}_a\boldsymbol{v}_b + 2\boldsymbol{v}_b^2 - \frac{3}{2}\left(\frac{\boldsymbol{v}_b\boldsymbol{x}_{ab}}{r_{ab}}\right)^2\Bigg] -$$

$$- \frac{7}{2}G\sum_{b\neq a}\left(\frac{m_b}{r_{ab}}\right)G\sum_{c\neq b,a}\frac{m_c\boldsymbol{x}_{bc}}{r_{bc}^3} + G\sum_{b\neq a} m_b\left(\frac{\boldsymbol{x}_{ab}}{r_{ab}^3}\right)(4\boldsymbol{v}_a - 3\boldsymbol{v}_b)\cdot(\boldsymbol{v}_a - \boldsymbol{v}_b). \tag{7.42}$$

7.2.1 Two Body Problem in the 1PN Approximation

The Einstein-Infeld-Hoffmann equations for the two body problem imply that the center of mass

$$\boldsymbol{X} = \frac{m_1^*\boldsymbol{x}_1 + m_2^*\boldsymbol{x}_2}{m_1^* + m_2^*} \tag{7.43}$$

with

$$m_a^* := m_a + \frac{1}{2}m_a\left(\frac{\boldsymbol{v}_a}{c}\right)^2 - \frac{1}{2}\frac{m_a m_b}{r_{ab}}\frac{G}{c^2} \qquad (a, b = 1, 2,\ a \neq b) \tag{7.44}$$

is not accelerated, i.e.

$$\frac{d^2\boldsymbol{X}}{dt^2} = 0. \tag{7.45}$$

We can choose $\boldsymbol{X} = \boldsymbol{0}$ such that

$$\boldsymbol{x}_1 = \left[\frac{m_2}{m} + \frac{\mu\delta m}{2m^2}\left(v^2 - \frac{m}{r}\right)\right]\boldsymbol{x} \tag{7.46}$$

$$\boldsymbol{x}_2 = \left[-\frac{m_1}{m} + \frac{\mu\delta m}{2m^2}\left(v^2 - \frac{m}{r}\right)\right]\boldsymbol{x} \tag{7.47}$$

where $\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$, $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2$, $m = m_1 + m_2$, $\delta m = m_1 - m_2$, $\mu = \frac{m_1 m_2}{m}$ (reduced mass).

For the relative motion we obtain from Eqs. (7.38), (7.39) with (7.46), (7.47) after dividing by μ:

$$\mathcal{L}_{\text{rel}} = \mathcal{L}_0 + \mathcal{L}_1 \tag{7.48}$$

with the Newtonian part $\mathcal{L}_0$ and the post-Newtonian perturbation $\mathcal{L}_1$:

$$\mathcal{L}_0 = \frac{1}{2}\boldsymbol{v}^2 + \frac{Gm}{r} \tag{7.49}$$

$$\mathcal{L}_1 = \frac{3}{8}\left(1 - \frac{3\mu}{m}\right)\frac{\boldsymbol{v}^4}{c^2} + \frac{Gm}{2rc^2}\left(3\boldsymbol{v}^2 + \frac{\mu}{m}\boldsymbol{v}^2 + \frac{\mu}{m}\left(\frac{\boldsymbol{v}\boldsymbol{x}}{r}\right)^2\right) - \frac{G^2m^2}{2r^2c^2}. \tag{7.50}$$

The corresponding Euler-Lagrange equation is $(c = 1)$

$$\dot{\boldsymbol{v}} = -\frac{Gm}{r^3}\boldsymbol{x}\left(1 - \frac{Gm}{r}\left(4 + \frac{2\mu}{m}\right) + \left(1 + \frac{3\mu}{m}\right)\boldsymbol{v}^2 - \left(\frac{3\mu}{2m}\right)\left(\frac{\boldsymbol{v}\boldsymbol{x}}{r}\right)^2\right) +$$

$$+ \frac{Gm}{r^3}\boldsymbol{v}(\boldsymbol{v}\boldsymbol{x})\left(4 - \frac{2\mu}{m}\right). \tag{7.51}$$

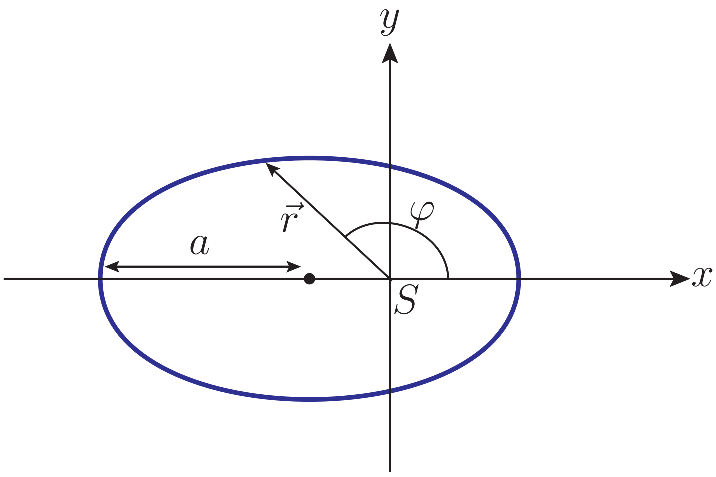

Figure 5: Elliptical Kepler orbit.

Consider the Kepler problem with motion in the plane $z = 0$ and the periastron lying on the x-axis. Without 1PN corrections one has an elliptic orbit (c.f. fig. 5) where e is the eccentricity and

$$r = \frac{a(1-e^2)}{1 + e\cos\varphi}. \tag{7.52}$$

One finds

$$r = a(1 - e\cos u) \qquad \text{where } u - e\sin u = \omega_0 t \tag{7.53}$$

(**Kepler's equation**). Here $\omega_0 = \frac{2\pi}{T}$ with orbital period T ($t = 0$ for passage at perihelion) and u is the so called **eccentric anomaly** ($\omega_0 t$ is the **mean anomaly**). We define $f = \varphi$, the **true anomaly**, such that (c.f. fig. 6)

$$\cos f = \cos\varphi = \frac{\cos u - e}{1 - e\cos u}. \tag{7.54}$$

Taking $\mathcal{L}_1$ from Eq. (7.50), we define

$$\varepsilon = \frac{E}{\mu} = \mathcal{L}_0 + \mathcal{L}_1 \tag{7.55}$$

and

$$\boldsymbol{j} = \frac{\boldsymbol{J}}{\mu} = \left[1 + \frac{1}{2}\left(1 - \frac{3\mu}{m}\right)\frac{v^2}{c^2} + \left(3 + \frac{\mu}{m}\right)\frac{Gm}{rc^2}\right](\boldsymbol{r} \wedge \boldsymbol{v}). \tag{7.56}$$

After some lengthy calculations (including an integration over time), one can get an equation which is analogous to Kepler's equation but with different coefficients:

$$\frac{2\pi}{T_b} t = u - e_t \sin u, \tag{7.57}$$

$$r = a_r(1 - e_r\cos u), \tag{7.58}$$

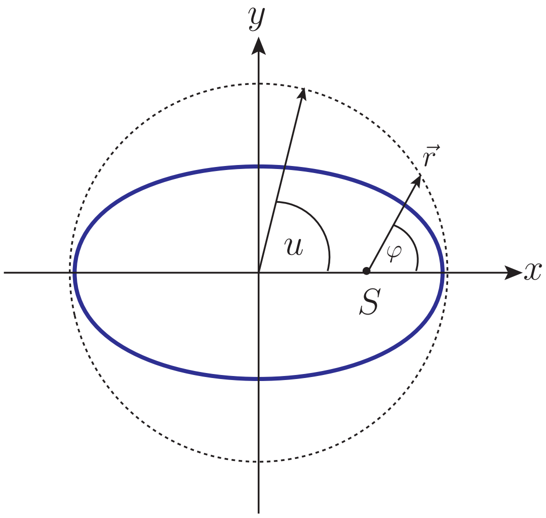

Figure 6: The variable u takes the role of the angle as measured from the center of the ellipse.

where

$$a_r = -\frac{Gm}{2\varepsilon}\left[1 - \left(\frac{\mu}{M} - 7\right)\frac{\varepsilon}{2c^2}\right] \tag{7.59}$$

$$e_r^2 = 1 + \frac{2\varepsilon}{G^2m^2}\left[1 + \left(5\frac{\mu}{m} - 15\right)\frac{\varepsilon}{2c^2}\right]\left[j^2 + \left(\frac{\mu}{m} - 6\right)\frac{G^2m^2}{c^2}\right] \tag{7.60}$$

$$e_t^2 = 1 + \frac{2\varepsilon}{G^2m^2}\left[1 + \left(17 - 7\frac{\mu}{m}\right)\frac{\varepsilon}{2c^2}\right]\left[j^2 + \left(2 - \frac{2\mu}{m}\right)\frac{G^2m^2}{c^2}\right] \tag{7.61}$$

$$\frac{2\pi}{T_b} = \frac{(-2\varepsilon)^{3/2}}{Gm}\left[1 - \left(\frac{\mu}{m} - 15\right)\frac{\varepsilon}{4c^2}\right], \tag{7.62}$$

with $j = |\boldsymbol{j}|$. Notice that, in a bound orbit, ε is negative. The eccentricity e of the Keplerian orbit is now split into a "radial eccentricity" e_r and a "time eccentricity" e_t. The Newtonian limit is found by considering $c \to \infty$:

$$a_r\big|_{c\to\infty} = \frac{Gm\mu}{-2E} \tag{7.63}$$

$$e_r^2\big|_{c\to\infty} = e_t^2\big|_{c\to\infty} = 1 + \frac{2EL^2}{G^2m^2\mu^3}, \tag{7.64}$$

where L simply denotes the Newtonian limit of $|\boldsymbol{J}|$, which is nothing but the classical angular momentum $L \equiv \mu\,|\boldsymbol{r} \wedge \boldsymbol{v}|$.

One finds for the true anomaly

$$\varphi(u) = \frac{\cos u - e_\theta}{1 - e_\theta \cos u} \qquad \text{or} \qquad \varphi(u) = A_{e_\theta}(u) = 2\arctan\left[\left(\frac{1+e_\theta}{1-e_\theta}\right)^{1/2}\tan\left(\frac{u}{2}\right)\right] \tag{7.65}$$

where $e_\theta = e$ for Kepler orbits. This yields

$$\varphi(u) = \omega_0 + (1+k)A_{e_\theta}(u) \qquad \text{with } k = \frac{3Gm}{c^2a(1-e^2)} \tag{7.66}$$

with $\omega_0^2 = (2\pi/T)^2 = Gm/a^3$. The quantity e_θ is called "angular eccentricity" and it satisfies

$$e_\theta^2 = 1 + \frac{2\varepsilon}{G^2m^2}\left[1 + \left(\frac{\mu}{m} - 15\right)\frac{\varepsilon}{2c^2}\right]\left[j^2 - \frac{6G^2m^2}{c^2}\right]. \tag{7.67}$$

For $c \to \infty$, we have to make the replacement $e_\theta \to e$.

One can also find the perihelion precession (or periastron precession) per orbit (c.f. section 25 of GR I). After some calculations one finds:

$$\boxed{\delta\varphi = \frac{6\pi G(m_1+m_2)}{c^2a(1-e^2)}.} \tag{7.68}$$

Note that a in this formula is the semi-major axis ($a(1-e^2) = p$); in section 25 of GR I it was half of the Schwarzschild radius, $a = \frac{Gm}{c^2}$. The relevant difference as compared to section 25 is the fact that we had only one mass (the Sun) in section 25. In (7.68) $m = m_1 + m_2$ is the sum of the two masses. In view of the non-linearities that are involved in the description of the system, this simple result is far from obvious.

For the binary pulsar PSR 1913+16 (c.f. section 6.2), the measured periastron shift is

$$\dot{\omega}_{\text{obs.}} = 4.226607 \pm 0.00007 \text{ deg yr}^{-1} \tag{7.69}$$

The GR prediction which follows from Eq. (7.68) and the known orbital element (given a period of about 7.75 hours, i.e. $\dot{\omega} \sim \delta\varphi \times 1130$ deg yr^{-1}) is

$$\dot{\omega}_{\text{GR}} = 2.11\left(\frac{m_1+m_2}{M_\odot}\right)^{2/3} \text{deg yr}^{-1} \tag{7.70}$$

where we used

$$a = \left[\frac{G}{\omega_0^2}(m_1+m_2)\right]^{1/3} \tag{7.71}$$

due to Kepler's third law. If we set $\dot{\omega}_{\text{obs.}} = \dot{\omega}_{\text{GR}}$, it follows $m_1 + m_2 = 2.83M_\odot$.

In order to characterize the pulsar system, several parameters are relevant:

- The parameters which characterize the pulsar itself: right ascension α, declination δ, proper motion, the initial pulse time ϕ_0, the pulse frequency ν, and the spindown parameter.
- The five Keplerian parameters (c.f. fig. 7): T_b (period), t_0 (time of passage at periastron), $x = a_r \sin i$ (i denotes the inclination angle of the orbital plane with respect to the observer), e (eccentricity), ω (angular position of periastron as measured from the ascending node).
- There are eight independent measurable post-Keplerian parameters (we state only the five main parameters):

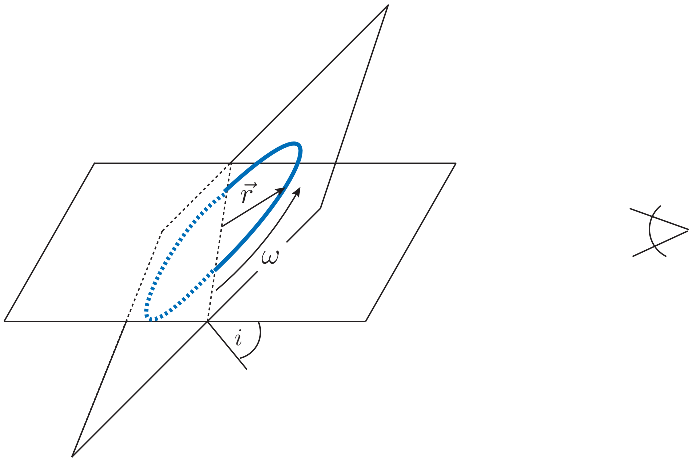

Figure 7: Sketch of spatial arrangement of post-Keplerian orbit relative to observer.

- $\dot{\omega}$ (periastron shift),
- γ (not to be confused with γ from Robertson expansion),
- Δ_{E} (Einstein time delay) which is related to the transformation from the pulsar proper time to the coordinate time of the pulsar-companion barycenter system. One finds

$$\Delta_{\mathrm{E}} = \gamma \sin u \tag{7.72}$$

$$\begin{aligned} \text{where } \gamma &= \left(\frac{T_b}{2\pi}\right)^{1/3} e \frac{G^{2/3}}{c^2} \frac{m_2(m_1 + 2m_2)}{a(m_1+m_2)} \\ &= 2.93696 \text{ ms} \cdot \left(\frac{m_2}{M_\odot}\right)\left(\frac{m_1+2m_2}{M_\odot}\right)\left(\frac{m_1+m_2}{M_\odot}\right)^{-4/3} \end{aligned} \tag{7.73}$$

  where m_2 is the mass of the companion and m_1 is the mass of the emitting pulsar.
- $r = \frac{Gm_2}{c^3}$ (Shapiro time delay) which corresponds to $\frac{a}{c} = \nu$ in Eq. (1.12),
- $\dot{T}_b$ see Eq. (6.38) due to emission of gravitational waves and depends also on the masses m_1 and m_2.
- $s = \sin i = cG^{-1/3}\left(\frac{T_b}{2\pi}\right)^{-2/3} m^{2/3} m_2^{-1}$
- $\delta_\theta = \frac{e_\theta - e_t}{e_t} = \frac{G}{c^2 am}\left(\frac{7}{2}m_1^2 + 6m_1m_2 + 2m_2^2\right)$ where we used Eqs. (7.67) and (7.61).

Seven parameters are needed to fully specify the dynamics of the two-body system (up to uninteresting rotation about the line of sight). Therefore, the measurement of any two post-Keplerian parameters (besides five Keplerian parameters) allows to predict the remaining ones. These parameters thus constitute a consistency check for GR.

8 The Kerr Solution

The Kerr solution is one of the most important solutions to Einstein's (vacuum) equations. It describes stationary rotating black holes, and was found in 1963 by R. Kerr[14]. Later, it was generalized to the **Kerr-Newman solution** which describes rotating, electrically charged black holes.

In this chapter we will give an overview over the Kerr solution and the Kerr-Newman solution without detailed calculations.

The Kerr solution is axisymmetric and stationary. We use the so-called **Boyer-Lindquist coordinates** (t, r, θ, φ) and we use the following abbreviations:

$$\Delta = r^2 - 2mr + a^2, \qquad \rho^2 = r^2 + a^2 \cos^2\theta \tag{8.1}$$

where we set $G = c = 1$. The Kerr metric reads

$$\boxed{\begin{aligned} ds^2 = \frac{1}{\rho^2}\Big[&-(\Delta - a^2\sin^2\theta)c^2\, dt^2 + 2a\sin^2\theta(\Delta - r^2 - a^2)c\, dtd\varphi + \\ &+ \sin^2\theta\left((r^2+a^2)^2 - \Delta a^2 \sin^2\theta\right) d\varphi^2\Big] + \rho^2\left[\frac{dr^2}{\Delta} + d\theta^2\right]. \end{aligned}} \tag{8.2}$$

8.1 Interpretation of the Parameters a and m

In order to interpret a and m we look at the asymptotic form of the metric (8.2) for large "radial coordinate" r:

$$\begin{aligned} ds^2 = &-\left[1 - \frac{2m}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right] dt^2 - \left[\frac{4am}{r}\sin^2\theta + \mathcal{O}\left(\frac{1}{r^2}\right)\right] dtd\varphi + \\ &+ \left[1 + \mathcal{O}\left(\frac{1}{r}\right)\right] dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \end{aligned} \tag{8.3}$$

The examination is easier by transforming to asymptotically Lorentzian coordinates (Cartesian coordinates):

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta. \tag{8.4}$$

This yields the following form of the asymptotic metric:

$$\begin{aligned} ds^2 = &-\left[1 - \frac{2m}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right] dt^2 - \left[\frac{4am}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right)\right] (x\, dy - y\, dx)dt + \\ &+ \left[1 + \mathcal{O}\left(\frac{1}{r}\right)\right] (dx^2 + dy^2 + dz^2), \end{aligned} \tag{8.5}$$

where we used $r^2\sin^2\theta\, d\varphi = x\, dy - y\, dx$.

In section 4 we computed the metric of the rotating Earth assuming slow rotation. In Eq. (4.18) we found

$$ds^2 = \left[1 - \frac{2GM_\mathrm{E}}{c^2 r}\right] c^2 dt^2 - \left[1 + \frac{2GM_\mathrm{E}}{c^2 r}\right] d\boldsymbol{r}^2 + 2ch_{0i}\, dx^i dt. \tag{8.6}$$

[14] Phys. Rev. Lett. **11**, 237 (1963)

Identifying $\frac{GM_E}{c^2} \to m$, $d\boldsymbol{r}^2 = dx^2 + dy^2 + dz^2$ and including an overall sign (since we used another sign convention in section 4), this metric should coincide with (8.5).

Using Eq. (4.1),

$$h_{0i} = \frac{4G}{c^3} \varepsilon_{ikn} \frac{\omega^k x^j}{r^3} \int d^3\tilde{r}\, \tilde{x}^n \rho(\tilde{\boldsymbol{r}}) \tilde{x}_j, \tag{8.7}$$

we infer that T^{0i} is proportional to $\rho \frac{v_i}{c}$ where $v_i = \varepsilon_{ikn} \omega^k x^n$ with ω^k being the angular velocity of the rotating body. We define

$$S_k = \varepsilon_{klm} \int d^3x\, x^l T^{m0} \tag{8.8}$$

which is the intrinsic angular momentum of the rotating body. Therefore,

$$h_{0i} = \frac{4G}{c^3} \varepsilon_{ikm} \frac{x^m S^k}{r^3}. \tag{8.9}$$

With the sign convention of (8.5), Eqs. (8.5) and (8.6) indeed take the common form

$$\begin{aligned} ds^2 = &- \left[1 - \frac{2m}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right] dt^2 + \left[1 + \frac{2m}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right] \delta_{ij}\, dx^i dx^j - \\ &- \left[4\varepsilon_{ikl} \frac{S^k x^l}{r^3} + \mathcal{O}\left(\frac{1}{r^4}\right)\right] dt dx^i. \end{aligned} \tag{8.10}$$

Clearly S^k is proportional to the body's mass M and its angular momentum (cf. Eq. (4.13)):

$$S^k = \frac{aGM}{c^2} \frac{\partial}{\partial z} = a \frac{GM}{c^2} \times \left(\begin{smallmatrix} \text{unit vector along polar axis} \\ \text{of Boyer-Lindquist coordinates} \end{smallmatrix} \right). \tag{8.11}$$

Therefore, m is just the mass, and a can be interpreted as the angular momentum ($0 \le a \le 1$).

8.2 Kerr-Newman Solution

The Kerr-Newman solution is the extension of the Kerr solution which also describes electrically charged black holes ($c = G = 1$). We use the three parameters

$$\Delta = r^2 - 2Mr + a^2 + Q^2 \tag{8.12}$$

$$\rho^2 = r^2 + a^2 \cos^2 \varphi \tag{8.13}$$

$$\Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \tag{8.14}$$

where M is the total mass, a the intrinsic angular momentum and Q the total charge. The metric coefficients of the Kerr-Newman metric are

$$\boxed{\begin{aligned} &g_{rr} = \frac{\rho^2}{\Delta}, \qquad g_{\theta\theta} = \rho^2, \qquad g_{\varphi\varphi} = \frac{\Sigma^2}{\rho^2} \sin\theta, \\ &g_{tt} = -1 + \frac{2Mr - Q^2}{\rho^2}, \qquad g_{t\varphi} = -a \frac{2Mr - Q^2}{\rho^2} \sin^2\theta \end{aligned}}$$

where we assumed $a > 0$ without loss of generality. This metric contains the following special cases:

- $Q = a = 0$: Schwarzschild solution,
- $a = 0$: Reissner-Nordstrøm solution,
- $Q = 0$: Kerr solution.

The electromagnetic field of the Kerr-Newman solution is

$$F = Q\rho^{-4}(r^2 - a^2\cos^2\theta)\, dr \wedge (dt - a\sin^2\theta\, d\varphi) + \\ + 2Q\rho^{-4} ar\cos\theta\sin\theta\, d\theta \wedge \big((r^2+a^2)d\varphi - a dt\big) \tag{8.15}$$

where $\wedge$ denotes the exterior product defined in section 15 of GR I .

From (8.16) one can deduce the asymptotic expressions for electric and magnetic fields (in r, θ, φ directions):

$$\begin{aligned} E_r &= F_{rt} = \frac{Q}{r^2} + \mathcal{O}\left(\frac{1}{r^3}\right) \\ \frac{E_\theta}{r} &= \frac{F_{\theta t}}{r} = \mathcal{O}\left(\frac{1}{r^4}\right) \\ \frac{E_\varphi}{r\sin\theta} &= \frac{F_{\theta t}}{r\sin\theta} = 0 \\ B_r &= \frac{F_{\theta\varphi}}{r^2\sin\theta} = \frac{2Qa}{r^3}\cos\theta + \mathcal{O}\left(\frac{1}{r^4}\right) \\ B_\theta &= \frac{F_{\varphi r}}{r\sin\theta} = \frac{Qa}{r^3}\sin\theta + \mathcal{O}\left(\frac{1}{r^4}\right) \\ B_\varphi &= \frac{F_{r\theta}}{r} = 0. \end{aligned} \tag{8.16}$$

We see immediately that asymptotically the electric field is a Coulomb field.

8.3 Equations of Motion for Test Particles

Let a test particle with electric charge e and rest mass μ move in the external fields of a Kerr-Newman black hole. The equations of motion are

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha{}_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = eF^\alpha{}_\beta\frac{dx^\beta}{d\lambda} \tag{8.17}$$

The best way to solve this equation turns out to be the Hamiltonian formalism.[15]

We can simplify the analysis by assuming that the metric is that of a Kerr black hole ($Q = 0$) and that the motion is confined to the equatorial plane ($\theta = 0$).[16] In this case, the metric has the following

[15] Details can be found in Straumann's book or the book by Misner, Thorne, Wheeler.

[16] Note that the equatorial plane of a rotating black hole is distinguished. When we solved the geodesic equations in the Schwarzschild background, any arbitrary plane which includes the origin was equivalent.

non-vanishing components:

$$g_{tt} = -\left(1 - \frac{2GM}{c^2 r}\right), \tag{8.18}$$

$$g_{t\varphi} = -\frac{a}{r}\frac{2GM}{c^2}, \tag{8.19}$$

$$g_{\varphi\varphi} = \left(r^2 + a^2 + \frac{a^2}{r}\frac{2GM}{c^2}\right). \tag{8.20}$$

Denoting by $K = \frac{E}{\mu c}$ the total energy and by l the angular momentum of the particle, one finds

$$\frac{1}{2}\mu\dot{r}^2 + \mu V_{\text{eff.}} = \text{const.} \tag{8.21}$$

with the effective potential

$$\begin{aligned} V_{\text{eff.}} &= -\frac{\frac{2GM}{c^2}c^2}{2r} + \frac{l^2 - a^2(K^2 - c^2)}{2\mu^2 r^2} - \frac{\frac{2GM}{c^2}(l - aK)^2}{2\mu^2 c^2 r^3} \\ &= -\frac{GM}{r} + \frac{l^2 - a^2(K^2 - c^2)}{2\mu^2 r^2} - \frac{GM(l - aK)^2}{\mu^2 c^2 r^3}. \end{aligned} \tag{8.22}$$

For $a = 0$ this reduces to the Schwarzschild case.

For a black hole which is spinning extremely fast, it can be shown that for a particle which spirals in towards the black hole in an accretion disc from very far away to the innermost circular stable orbit, the fraction $\left(1 - \frac{1}{\sqrt{3}}\right)$ of its rest energy is set free. The innermost stable circular orbit can easily be determined from Eq. (8.22). Thus a rotating black hole allows a gravitational energy conversion with an efficiency up to $\approx 42.3\%$!

These considerations are of astrophysical importance since quasars in the center of galaxies are supermassive rotating black holes.

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