MMP I
Solution Sheet 9

HS 21
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Exercise 1 [Hanging chain (6 points)]

a) length: $G[y] = 2l = \int_{s(-L)}^{s(L)} ds = \int_{-L}^{L} \sqrt{dx^2 + dy^2} = \int_{-L}^{L} \sqrt{1+y'^2}dx$

potential: $\Delta V = \mu g y(s) \Delta s$, $m = \mu \Delta s$

$\Rightarrow V[y] = \int_{s(-L)}^{s(L)} \mu g y(s) ds = \mu g \int_{-L}^{L} y\sqrt{1+y'^2}dx$

Introduce a Lagrange multiplier:

$V - \lambda G = \int_{-L}^{L} \underbrace{\sqrt{1+y'^2}(\mu g y - \lambda)}_{F(y,y')} dx$ has to be minimized

From special case (ii) in the lecture with $F_x = 0$ we know that the first integral is

$F - y' F_{y'} = const \equiv C.$

$$\Rightarrow \sqrt{1+y'^2}(\mu g y - \lambda) - y' \frac{y'}{\sqrt{1+y'^2}}(\mu g y - \lambda) = C \qquad | \cdot \sqrt{1+y'^2}$$

$$\mu g y - \lambda = C\sqrt{1+y'^2} \qquad |^{\wedge 2}$$

$$1 + y'^2 = \left(\frac{\mu g y - \lambda}{C}\right)^2$$

$$\Rightarrow y'(x) = \pm\sqrt{\left(\frac{\mu g y - \lambda}{C}\right)^2 - 1}$$

$$\text{(Separation)} \Rightarrow \frac{dy}{\pm\sqrt{\left(\frac{\mu g y-\lambda}{C}\right)^2 - 1}} = dx$$

$$\Rightarrow \pm \int \frac{dy}{\sqrt{\left(\frac{\mu g y-\lambda}{C}\right)^2 - 1}} = \int dx = x + \beta$$

With $z = \frac{\mu g y - \lambda}{C}$ and $dz = \frac{\mu g}{C} dy$:

$$\int \frac{dy}{\sqrt{\left(\frac{\mu g y-\lambda}{C}\right)^2 - 1}} = \frac{C}{\mu g}\int \frac{dz}{\sqrt{z^2-1}}$$

With $z = \cosh\varphi$, $\cosh^2\varphi - \sinh^2\varphi = 1$ and $dz = \sinh\varphi d\varphi$:

$$\frac{C}{\mu g}\int \frac{dz}{\sqrt{z^2-1}} = \frac{C}{\mu g}\int \frac{\sinh\varphi d\varphi}{\sinh\varphi} = \frac{C}{\mu g}\varphi$$

$$\begin{aligned}
&\Rightarrow \pm\frac{C}{\mu g}\varphi = x + \beta \\
&\Rightarrow \frac{C}{\mu g}\operatorname{arcosh} z = \pm(x+\beta) \\
&\Rightarrow z = \cosh\left(\frac{\mu g}{C}(x+\beta)\right) \\
&\Rightarrow y = \frac{C}{\mu g}\cosh\left(\frac{\mu g}{C}(x+\beta)\right) + \frac{\lambda}{\mu g}
\end{aligned}$$

b)

$$\begin{aligned}
y(x) = y(-x) &\Rightarrow \cosh\left(\frac{\mu g}{C}(x+\beta)\right) = \cosh\left(\frac{\mu g}{C}(-x+\beta)\right) \\
&\Rightarrow \beta = 0 \\
&\Rightarrow y(x) = \frac{C}{\mu g}\cosh\left(\frac{\mu g}{C}x\right) + \frac{\lambda}{\mu g}
\end{aligned}$$

c)

$$\begin{aligned}
y'(x) &= \sinh\left(\frac{\mu g}{C}x\right) \\
\sqrt{1+y'^2} &= \cosh\left(\frac{\mu g}{C}x\right)
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \int_{-L}^{L}\sqrt{1+y'^2}dx &= 2\int_0^L \cosh\left(\frac{\mu g}{C}x\right) = 2\frac{C}{\mu g}\sinh\left(\frac{\mu g}{C}x\right)\Big|_0^L \\
&= 2\frac{C}{\mu g}\sinh\left(\frac{\mu g L}{C}\right) \overset{!}{=} 2l \\
\Rightarrow \sinh(k) &= k\cdot\frac{l}{L} \\
k &:= \frac{\mu g L}{C} \\
\frac{\sinh k}{k} &= \frac{l}{L} = const.
\end{aligned}$$

d) $y(+L) = h = \frac{C}{\mu g}\cosh\left(\frac{\mu g L}{C}\right) + \frac{\lambda}{\mu g}$

$$\Rightarrow \lambda(C) = h\mu g - C\cosh\left(\frac{\mu g L}{C}\right)$$
$$\lambda(k) = h\mu g - \frac{\mu g L}{k}\cosh(k)$$

Exercise 2 [Loop (3 points)]

Note that this exercise was more involved then intended and arguably not solvable following the hint. The 3 points are given as bonus points.

We consider a piecewise continuously differentiable curve

$$\gamma : [a, b] \to \mathbb{R}^2, \quad t \mapsto \gamma(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}.$$

Its length is given by the functional

$$\mathcal{L}[\gamma] = \int_a^b \mathrm{d}s = \int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\mathrm{d}t. \tag{1}$$

Starting from the boundary condition we know that $\mathcal{L}[\gamma] = L = \text{const.} > 0$.

We want to find the closed curve γ of length L which maximises the area around a point. In order to do so, we first require parametrization of the area in terms of the curve parameter t. This parametrization can be obtained in two ways: (i) By applying the Stokes theorem to a two-dimensional surface; (ii) Through an infinitesimal consideration:

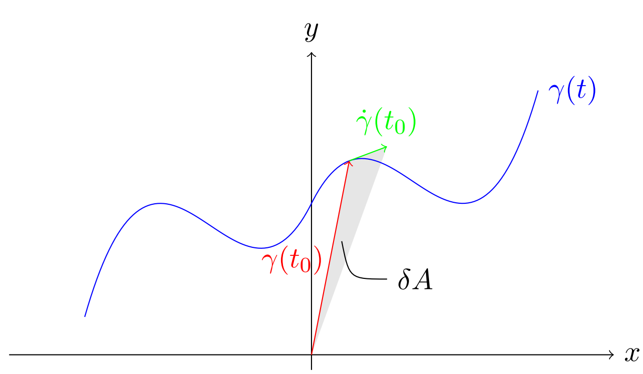

Figure 1: Pictorial derivation of the area element parametrized by the curve parameter

The area of a triangle spanned by two vectors is given by the cross product through $A = \frac{1}{2}|\vec{v}_1 \times \vec{v}_2|$. Considering the area element in Fig. 1 we see that the area $\delta A = \dfrac{\mathrm{d}A}{\mathrm{d}t}$ of the infinitesimal triangle is then given by

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \delta A \cdot \vec{n} = \frac{1}{2}\gamma(t) \times \dot{\gamma}(t) = \frac{1}{2}\left(\begin{pmatrix} x(t) \\ y(t) \\ 0 \end{pmatrix} \times \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ 0 \end{pmatrix}\right) \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{2}(x\dot{y} - y\dot{x})(t), \tag{2}$$

which is a locally invertible, piecewise continuously differentiable function.

By this we can express our area in terms of the curve parameter t by

$$\mathcal{A}[\gamma] = \int \mathrm{d}A = \int_a^b \frac{1}{2}(x\dot{y} - y\dot{x})(t)\,\mathrm{d}t. \tag{3}$$

Introducing a Lagrange parameter λ, we need to find the stationary solution of

$$\mathcal{A}[\gamma] - \lambda\mathcal{L}[\gamma] = \int_a^b \underbrace{\left[\frac{1}{2}(x\dot{y} - y\dot{x})(t) - \lambda\sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\right]}_{=F(x,\dot{x},y,\dot{y},\not{t})} \mathrm{d}t. \tag{4}$$

The stationary solutions are given by the Euler-Lagrange equations

$$\frac{\partial F}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial \dot{x}} = 0 \qquad \text{and} \qquad \frac{\partial F}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial F}{\partial \dot{y}} = 0. \tag{5}$$

In our case they are given by

$$\begin{aligned} \frac{\dot{y}}{2} - \frac{\mathrm{d}}{\mathrm{d}t}\left(-\frac{y}{2} - \lambda\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) &= 0, \\ -\frac{\dot{x}}{2} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{x}{2} - \lambda\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) &= 0. \end{aligned}$$

By integrating, squaring and adding up the two equations one obtains

$$\lambda^2 = (x - x_0)^2 + (y - y_0)^2$$

which is the equation of a circle centered around (x_0, y_0).

Another way to solve this problem and obtain explicit solutions for x and y is the following: We parametrize the curve by arc length, that is (w.l.o.g. $\mathrm{d}t/\mathrm{d}s > 0$)

$$\begin{aligned} 1 = \left|\left|\frac{\mathrm{d}\tilde{\gamma}'(s)}{\mathrm{d}s}\right|\right| &= \left|\frac{\mathrm{d}t}{\mathrm{d}s}\right| \cdot ||\dot{\gamma}(t(s))|| \\ \Leftrightarrow \tilde{x}'(s) = \frac{\mathrm{d}t}{\mathrm{d}s} \cdot \dot{x}(t(s)), \qquad \tilde{y}'(s) &= \frac{\mathrm{d}t}{\mathrm{d}s} \cdot \dot{y}(t(s)) \end{aligned}$$

From this the first Euler-Lagrange equation becomes

$$\frac{\mathrm{d}s}{\mathrm{d}t}\frac{y'(s)}{2} - \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{y(s)}{2} - \lambda\frac{\frac{\mathrm{d}s}{\mathrm{d}t}x'(s)}{\sqrt{\left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 x'(s)^2 + \left(\frac{\mathrm{d}s}{\mathrm{d}t}\right)^2 y'(s)^2}}\right) = 0$$

$$\overset{\dot{s}>0}{\Leftrightarrow} 0 = \frac{y'(s)}{2} - \frac{\mathrm{d}}{\mathrm{d}s}\left(-\frac{y(s)}{2} - \lambda x'(s)\underbrace{\frac{1}{\sqrt{x'(s)^2 + y'(s)^2}}}_{=1}\right)$$

and equally for the second equation.

By evaluating the derivative we obtain the second order differential equations

$$y' + \lambda x'' = 0, \qquad -x' + \lambda y'' = 0,$$

which are solved by

$$x(s) = x_0 + \lambda\cos\frac{s - s_0}{\lambda}, \qquad y(s) = y_0 + \lambda\sin\frac{s - s_0}{\lambda},$$

and parametrize a circle of radius λ.

Inserting this parametrization into $L = \mathcal{L}[(x, y)]$ and integrating over one revolution of the circle yields the condition

$$L = \int_\circ \mathrm{d}s = 2\pi\lambda \quad \Rightarrow \lambda = \frac{L}{2\pi} \tag{6}$$

fixes the Lagrange parameter. By this we see that the curve of length L maximizing the area around a point is a circle of radius $L/2\pi$.

Exercise 3 [Geometrical optics (5 points)]

a) $n := \frac{c_0}{c_m}$; c_0 = speed of light in vacuum, c_m = speed of light in medium

$T[\vec{x}] = \int\limits_{P_1}^{P_2} \frac{ds}{v(x,y,z)} = \frac{1}{c_0}\int\limits_{P_1}^{P_2} n(x, y, z)\sqrt{dx^2 + dy^2 + dz^2}$

$T[\vec{x}]$ = total travel time, $v = c_m = \frac{c_0}{n}$, $ds^2 = dx^2 + dy^2 + dz^2$ (Euclidean metric)

$\rightarrow$ Since we integrate along a curve, we can introduce an affine parameter λ :

$T[\vec{x}] == \frac{1}{c_0}\int\limits_{\lambda_1}^{\lambda_2} \underbrace{n(x(\lambda), y(\lambda), z(\lambda))\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}_{\equiv J(\lambda,\vec{x},\dot{\vec{x}}) = J(\lambda,x,y,z,\dot{x},\dot{y},\dot{z})} d\lambda$

$\Rightarrow$ 3 Euler equations: $J_{x_i} - \frac{d}{d\lambda}J_{\dot{x}_i} = 0$, $i = 1, 2, 3$

$\frac{\partial J}{\partial x_i} = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\frac{\partial n}{\partial x_i}, \; \frac{d}{d\lambda}J_{\dot{x}_i} = \frac{d}{d\lambda}n(x,y,z)\frac{\dot{x}_i}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}$

$\Rightarrow \frac{\partial n}{\partial x_i} = \frac{d}{\underbrace{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}d\lambda}_{ds}}\left[n(x,y,z)\frac{dx_i}{\underbrace{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}d\lambda}_{ds}}\right]$

$\Rightarrow \vec{\nabla}n = \frac{d}{ds}\left[n\frac{d\vec{x}}{ds}\right]$

b) $\cos\alpha = \frac{dy}{ds}$, $\sin\alpha = \frac{dx}{ds}$

$\frac{\partial n}{\partial x} = 0 = \frac{d}{ds}\left(n\frac{dx}{ds}\right) = \frac{d}{ds}(n\sin\alpha)$

$\Rightarrow n\sin\alpha = const.$

c) $n_1\sin\alpha_1 = const.$, $n_2\sin\alpha_2 = const.$

The point of refraction is not fixed, but the path must be continuous at $y = 0$. (natural boundary condition)

Choose a parametrisation $x(\lambda)$, $y(\lambda)$ such that $y(0) = 0$

$$\begin{aligned} 0 = \lim_{\varepsilon\to 0}[J_{\dot{x}}]_{\lambda=-\varepsilon}^{\lambda=\varepsilon} &= \lim_{\varepsilon\to 0}\left[n(x,y)\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right]_{\lambda=-\varepsilon}^{\lambda=\varepsilon} \\ &= \lim_{\varepsilon\to 0}\left[n(x,y)\sin\alpha\right]_{\lambda=-\varepsilon}^{\lambda=\varepsilon} \\ &= n_1\sin\alpha_1 - n_2\sin\alpha_2 \\ &\Rightarrow n_1\sin\alpha_1 = n_2\sin\alpha_2 \end{aligned}$$