1.5 Renormalization and Resummation

The concepts of effective field theories and renormalization are closely connected to each other. First, we will find that through the renormalization of the effective theory, we will be able to solve to problem of the large logarithms that we briefly touched upon earlier. In the end, the idea of EFTs will give a modern perspective on renormalization in general.

1.5.1 Renormalizability

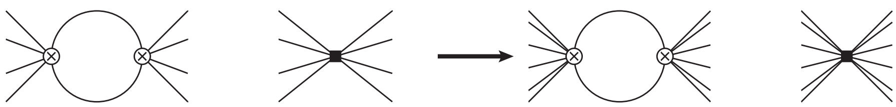
While we have discussed loops and even seen divergences in the matching. In order to remove these divergences, the theory needs to be renormalized. However, our infinite sum of operator products in the OPE will contain operators with mass dimension $k > 4$ and they are thus not renormalizable in the classical sense. First, let us get some more terminology in place. In our OPE, each operator's contribution to the action is (for a process with energies $E \ll \Lambda$):

$$\delta S_{(i,k)} \sim \mathcal{C}_{(i,k)} \left(\frac{E}{\Lambda}\right)^{k-4} , \tag{1.1}$$

with k being the operator dimension. Operators are then classified by their importance at low energies ($E \to 0$):

k	low-energy behavior	classical renormalizability	name
< 4	grows	super-renormalizable	*relevant*
$= 4$	constant	renormalizable	*marginal*
> 4	decreases	non-renormalizable	*irrelevant*

In our OPE, we allowed for operators of dimension $k > 4$, which are not renormalizable in the classical sense, meaning that we would need an infinite number of counterterms to cancel the divergences. This is easily seen by inserting a non-renormalizable operator twice in a loop-graph:



For example taking an operator of the form $\Delta\mathcal{L}_{\text{eff}} = g\varphi^6/\Lambda^2$, we can draw a diagram with two insertions of this operator and connecting four legs to form a loop-graph. This graph will have $2\times6-4 = 8$ external legs and will be divergent. This divergence needs to be absorbed into a counterterm of a φ^8 operator. This in turn can be used to generate a divergent 10-point amplitude which introduces higher operators and so on.

This apparent problem with our effective operators is cured once we implement power-counting: The divergence from inserting the φ^6 twice gives an amplitude of

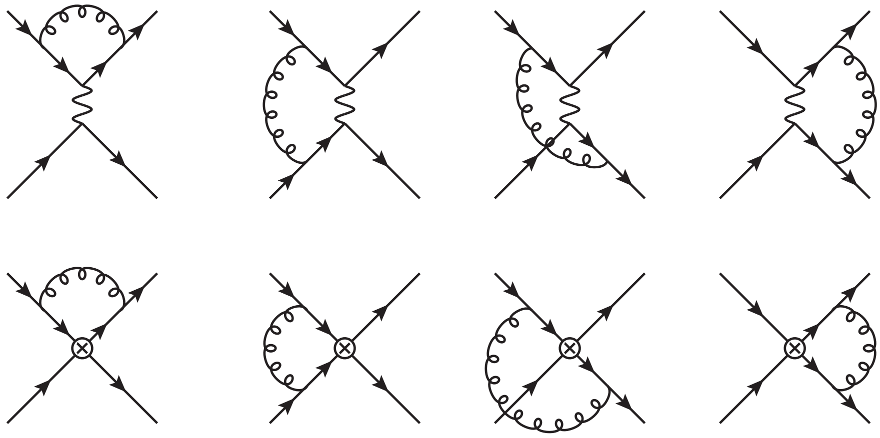
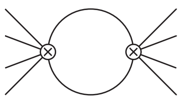


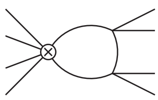
Figure 1.1: Example diagrams for the one-loop matching of the Fermi-theory. The top row depicts diagrams in the full SM whereas the bottom row shows corresponding diagrams in the EFT.

the form:



$$\sim \frac{g^2}{\Lambda^4}\int \frac{d^d l}{(2\pi)^d}\left(\frac{1}{l^2}\right)^2 = \frac{g^2}{\Lambda^4}\cdot\left(\frac{i}{16\pi^2\epsilon}+\mathcal{O}(\epsilon^0)\right)\,, \tag{1.2}$$

so it is of higher order in power-counting than the operators we inserted. Thus, if we decide to stop our OPE at dimension-six, then this correction is dropped. The only graph that does indeed contribute at this order in power-counting is the one where instead of the second operator insertion we insert a (renormalizable) quartic coupling twice. But this graph has more propagators, regulating it in the UV and rendering it finite:



$$\sim \frac{g\lambda^2}{\Lambda^2}\int \frac{d^d l}{(2\pi)^d}\left(\frac{1}{l^2}\right)^3 = \mathcal{O}(\epsilon^0)\,. \tag{1.3}$$

Therefore, once we stick consequently to our power-counting, our theory contains only a finite amount of counterterms and is thus renormalizable, in the modern sense.

1.5.2 Resummation of large logarithms

Now that we know the effective theory can be renormalized, we should discuss the renormalization in detail. We will do this at a realistic example, the quark transition $b \to c\bar{u}d$. In the full SM, this transition occurs through the weak decay of the b-quark through a virtual W-boson. Again, the momentum flow through the W-boson propagator is small compared to the mass $q^2 \sim m_b^2 \ll m_W^2$, so we can match it onto a set of effective operators obtained after integrating out the W. The tree-level contribution receives important corrections at one-loop order in QCD. Evaluating graphs like the ones in the top row of Fig. 1.1, we find the full $\mathcal{O}(\alpha_s)$-result:

$$\begin{aligned} i\mathcal{A}_{\text{full}} = -iG \Bigg\{ &(\bar{u}(p_c)\gamma^\mu P_L u(p_b))(\bar{u}(p_d)\gamma_\mu P_L v(p_u)) \left[1 - \frac{\alpha_s}{4\pi}\left(\log\frac{q^2}{m_W^2} + \frac{3}{2}\right)\right] \\ &+(\bar{u}(p_c)\gamma^\mu P_L v(p_u))(\bar{u}(p_d)\gamma_\mu P_L u(p_b)) \left[\frac{3\alpha_s}{4\pi}\left(\log\frac{q^2}{m_W^2} + \frac{3}{2}\right)\right]\Bigg\} , \end{aligned} \tag{1.4}$$

with $G = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^*$. Note the large logarithm of the scale ratio q^2/m_W^2, rendering our perturbation series poorly convergent. Higher-order corrections will introduce terms of the form $\alpha_s^k \log^k(q^2/m_W^2)$, which will be large as well. We would need to go to very high loop-order to get a result with small perturbative uncertainty.

It would be desirable if we could introduce a new counting scheme: Instead of doing perturbation theory in α_s, we could introduce a parameter η and assign the counting

$$\alpha_s \sim \eta\,, \qquad \log\frac{q^2}{m_W^2} \sim \frac{1}{\eta}\,. \tag{1.5}$$

If we could now find a way to rearrange the perturbation series from an expansion in α_s into an expansion in η, then the resulting series would be more convergent. To resum the perturbation series in η, we would need the logarithmic terms to all orders in perturbation theory, meaning we would have to compute an infinite number of loop graphs.

The way out is to utilize the large separation between q^2 and m_W^2, integrate out the W and match it to a basis of operators. By the argument of regions, the Wilson coefficients can only be functions of the hard scale. Defining the effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \mathcal{C}_1(\mu)\mathcal{O}_1 + \mathcal{C}_2(\mu)\mathcal{O}_2\,, \tag{1.6}$$

with the effective operators

$$\mathcal{O}_1 = (\bar{c}_L\gamma^\mu b_L)(\bar{d}_L\gamma_\mu u_L)\,, \qquad \mathcal{O}_2 = (\bar{c}_L\gamma^\mu u_L)(\bar{d}_L\gamma_\mu b_L)\,, \tag{1.7}$$

we can indeed evaluate the full theory diagrams once more in the hard region. In this case, we have:

$$\begin{aligned} i\mathcal{A}_{\text{full}}^{\text{hard}} = -iG \Bigg\{ &(\bar{u}(p_c)\gamma^\mu P_L u(p_b))(\bar{u}(p_d)\gamma_\mu P_L v(p_u)) \\ &\times\left[1 - \frac{\alpha_s}{4\pi}\left(\frac{11}{3}\frac{1}{\epsilon_{\text{IR}}} + \log\frac{\mu_{\text{IR}}^2}{m_W^2} + \frac{8}{3}\log\frac{\mu_{\text{IR}}^2}{\mu_{\text{UV}}^2} + \frac{3}{2}\right)\right] \\ &+(\bar{u}(p_c)\gamma^\mu P_L v(p_u))(\bar{u}(p_d)\gamma_\mu P_L u(p_b)) \left[\frac{3\alpha_s}{4\pi}\left(\frac{1}{\epsilon_{\text{IR}}} + \log\frac{\mu_{\text{IR}}^2}{m_W^2} + \frac{3}{2}\right)\right]\Bigg\} , \end{aligned} \tag{1.8}$$

which we can directly match onto the Lagrangian (1.6). Here we have assigned indices IR and UV not only to the dimensional regulators ϵ_i, but also to the corresponding renormalization scale μ_i. The $1/\epsilon_{\text{UV}}$ terms are cancelled by the QCD counterterms, as discussed earlier. The matching yields:

$$\begin{aligned} \mathcal{C}_1(\mu) &= -G\left\{1 - \frac{\alpha_s(\mu)}{4\pi}\left(\frac{11}{3}\frac{1}{\epsilon_{\text{IR}}} + \log\frac{\mu_{\text{IR}}^2}{m_W^2} + \frac{8}{3}\log\frac{\mu_{\text{IR}}^2}{\mu_{\text{UV}}^2} + \frac{3}{2}\right)\right\} , \\ \mathcal{C}_2(\mu) &= -G\left\{\frac{3\alpha_s(\mu)}{4\pi}\left(\frac{1}{\epsilon_{\text{IR}}} + \log\frac{\mu_{\text{IR}}^2}{m_W^2} + \frac{3}{2}\right)\right\} . \end{aligned} \tag{1.9}$$

We are still keeping the renormalization scales $\mu_{\rm IR}$ and $\mu_{\rm UV}$ separate. When comparing the amplitudes (1.4) and (1.8), we see that the logarithmic dependence on q^2 has been turned into an IR divergence, $\log q^2 \to 1/\epsilon + \log \mu^2$. This makes sense, since the loop diagrams in the hard region lack the IR regulators that rendered the full theory result finite. The Wilson coefficients therefore now contain new divergences, associated to the fact that we separated the theory into a UV and an IR piece.

The logarithm that the coefficients now contain can either be large or small, depending on the scale μ at which the Wilson coefficients are evaluated. And Evaluating the Wilson coefficients at the natural scale of the process $\mu^2 = q^2$ restores the full theory result (up to the $1/\epsilon$-term, which can be absorbed into a counterterm). However, we can evaluate the Wilson coefficients at a different scale $\mu^2 \neq q^2$ and use the renormalization group (RG) to take it to the scale $\mu^2 = q^2$. This is the key to solving the problem of large logarithms: At the scale $\mu^2 = m_W^2$, the large logarithms are exactly zero. We then set up and solve the differential equations obtained by imposing invariance under the change of μ - the RG equation (RGEs) - and obtain the correct result at the low scale. Since at $\mu^2 = m_W^2$, the expression trivially contains all orders of $\alpha_s^k \log^k(m_W^2/m_W^2)$, solving the RGEs to get to the low scale will give us a result that also contains the large logarithms $\alpha_s^k \log^k(q^2/m_W^2)$. We say that the RG running resums the large logarithms.

Let us now return to our example and demonstrate the resummation explicitely. First, we renormalize our Lagrangian in the usual way, rescaling the fields:

$$\psi \to \sqrt{Z_\psi}\psi\,, \tag{1.10}$$

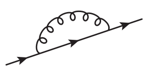
then we have:

$$\mathcal{L}_{\rm eff} = Z_\psi^2 \mathcal{C}_1^{(0)} \mathcal{O}_1 + Z_\psi^2 \mathcal{C}_2^{(0)} \mathcal{O}_2\,. \tag{1.11}$$

Now we assign

$$\mathcal{C}_i(1+\delta_{\mathcal{C}_i}) = Z_\psi^2 \mathcal{C}_i^{(0)}\,, \tag{1.12}$$

to find our counterterms. First, we need the wavefunction renormalization of the quarks to $\mathcal{O}(\alpha_s)$. It is a straightforward computation and yields:



$$\begin{aligned} &= (-4\pi\alpha_s C_F)\int \frac{d^dl}{(2\pi)^d}\frac{\gamma^\mu(\not{l}+\not{p})\gamma_\mu}{l^2(l+p)^2} = i\not{p}\,\frac{\alpha_s C_F}{4\pi\epsilon}\,, \\ &\Rightarrow Z_\psi = 1 - \frac{\alpha_s(\mu)}{3\pi\epsilon}\,. \end{aligned} \tag{1.13}$$

Next, we need the counterterms of the effective couplings $\mathcal{C}_i$. To this end, we need to compute the divergences of the matrix elements of the EFT, corresponding to graphs like the ones in the bottom row of Fig. 1.1. We find:

$$\begin{aligned} i\mathcal{A}_{\rm EFT}^{(UV)} = i\Bigg\{ &(\bar{u}(p_c)\gamma^\mu P_L u(p_b))(\bar{u}(p_d)\gamma_\mu P_L v(p_u))\mathcal{C}_1(\mu)\left(1+\frac{\alpha_s(\mu)}{4\pi\epsilon_{\rm UV}}\left[\frac{11}{3}-3\frac{\mathcal{C}_2(\mu)}{\mathcal{C}_1(\mu)}\right]\right) \\ &+ (\bar{u}(p_c)\gamma^\mu P_L v(p_u))(\bar{u}(p_d)\gamma_\mu P_L u(p_b))\mathcal{C}_2(\mu)\left(1+\frac{\alpha_s(\mu)}{4\pi\epsilon_{\rm UV}}\left[\frac{11}{3}-3\frac{\mathcal{C}_1(\mu)}{\mathcal{C}_2(\mu)}\right]\right)\Bigg\}\,. \end{aligned} \tag{1.14}$$

Notice that once we insert (1.9) into this expression, the IR divergences of the bare Wilson coefficients cancel the UV divergences of the matrix element. This is because the product of both is an object that contains both the hard and soft region, which is the full result and thus finite (after the UV theory is renormalized).

We can read off the counterterms for the Wilson coefficients from eq. (1.14), they are

$$\begin{aligned}\delta_{\mathcal{C}_1} &= -\frac{\alpha_s(\mu)}{\pi\epsilon}\left(\frac{11}{12}-\frac{3\mathcal{C}_2(\mu)}{4\mathcal{C}_1(\mu)}\right)+\mathcal{O}(\alpha_s^2)\,,\\ \delta_{\mathcal{C}_2} &= -\frac{\alpha_s(\mu)}{\pi\epsilon}\left(\frac{11}{12}-\frac{3\mathcal{C}_1(\mu)}{4\mathcal{C}_2(\mu)}\right)+\mathcal{O}(\alpha_s^2)\,.\end{aligned} \tag{1.15}$$

Note that the divergence of $\mathcal{C}_2$ is proportional to $\mathcal{C}_1$ and vice versa. So even when there is no tree-level contribution to $\mathcal{C}_2$, there is a loop-induced contribution. We say $\mathcal{C}_2$ is *radiatively generated.*

With all this in place, we can write down the RG equations for the coefficients and solve them. First, demand that the bare coefficients $\mathcal{C}_i^{(0)}$ are independent of μ:

$$\frac{d\mathcal{C}_i^{(0)}}{d\log\mu} = \mu\frac{d}{d\mu}\left(\mathcal{C}_i\frac{Z_{\mathcal{C}_i}}{Z_\psi^2}\right) \overset{!}{=} 0\,. \tag{1.16}$$

By using the fact that the counterterms depend on the scale only through α_s, the equation can be rearranged into

$$\begin{aligned}\mu\frac{d}{d\mu}\mathcal{C}_i &= -\frac{Z_\psi^2}{Z_{\mathcal{C}_i}}\mathcal{C}_i\left(\mu\frac{d\alpha_s}{d\mu}\right)\frac{d}{d\alpha_s}\left(\frac{Z_{\mathcal{C}_i}}{Z_\psi^2}\right)\\ \mu\frac{d}{d\mu}\mathcal{C}_i &= -\mathcal{C}_i\left(-2\alpha_s\epsilon\right)\frac{d}{d\alpha_s}\left(\delta Z_{\mathcal{C}_i}-2\delta Z_\psi\right)\end{aligned} \tag{1.17}$$

In the second line we have inserted the leading term in the QCD beta function and dropped the prefactors since they differ from unity only at $\mathcal{O}(\alpha_s)$. Now we can insert the counterterms and get the evolution equations. It makes sense to define a vector $\vec{\mathcal{C}}(\mu) = (\mathcal{C}_1(\mu), \mathcal{C}_2(\mu))$ and write the equation in the compact form:

$$\mu\frac{d}{d\mu}\vec{\mathcal{C}}(\mu) = \frac{\alpha_s(\mu)}{2\pi}\begin{pmatrix}-1 & 3\\ 3 & -1\end{pmatrix}\cdot\vec{\mathcal{C}}(\mu) \tag{1.18}$$

With the QCD beta function $d\alpha_s(\mu)/d\log\mu = -\alpha_s^2(\mu)\beta_0/(2\pi)$, this equation becomes

$$\frac{d}{d\alpha_s}\vec{\mathcal{C}}(\alpha_s) = -\frac{1}{\alpha_s\beta_0}\begin{pmatrix}-1 & 3\\ 3 & -1\end{pmatrix}\cdot\vec{\mathcal{C}}(\alpha_s)\,. \tag{1.19}$$

The solution is given by:

$$\vec{\mathcal{C}}(\mu_l) = V\begin{pmatrix}R^{-4/\beta_0} & 0\\ 0 & R^{2/\beta_0}\end{pmatrix}V^{-1}\vec{\mathcal{C}}(\mu_h) \tag{1.20}$$

with $R = \alpha_s(\mu_h)/\alpha_s(\mu_l)$ and

$$V = \begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\,. \tag{1.21}$$

We now know the Wilson coefficients at the low scale μ_l as a function of the coefficients at the high scale μ_h. To see that this solution does indeed resum an infinite number of the logarithms $[\alpha_s(\mu_l)\log(q^2/m_W^2)]^k \equiv L^k$, we solve the QCD beta function at one-loop order:

$$\mu\frac{d\alpha_s(\mu)}{d\mu} = -\frac{\alpha_s^2}{2\pi}\beta_0 \qquad \Rightarrow \qquad \alpha(\mu_h) = \frac{\alpha(\mu_l)}{1-\frac{\alpha(\mu_l)}{4\pi\beta_0}\log\frac{\mu_l^2}{\mu_h^2}}\,. \tag{1.22}$$

Then we can express our variable R in terms of $\alpha_s(\mu_l)$ to find:

$$R = \frac{1}{1-\frac{\beta_0}{4\pi}\alpha_s(\mu_l)\log\frac{\mu_l^2}{\mu_h^2}} \tag{1.23}$$

Now seting $\mu_h^2 = m_W^2$ and $\mu_l^2 = q^2$, we see that the logarithms in R are exactly the logarithms L, that we want to resum and that the solution (1.20) indeed is an infinite tower of these logarithms:

$$R^b = \sum_{k=0}^{\infty} L^k \left(\frac{\beta_0}{4\pi}\right)^k \frac{\Gamma(b+k)}{\Gamma(b)\Gamma(1+k)}\,. \tag{1.24}$$

We say *the renormalization group evolution has resummed the large logarithms.* To see that these really correspond to the large logarithms we saw in eq. (1.4), we can simply stop the sum at $k=1$. Then we find:

$$\begin{aligned} \mathcal{C}_1(q) &= \mathcal{C}_1(m_W)\left(1-\frac{\alpha_s(q)}{4\pi}\log\frac{q^2}{m_W^2}\right)\,, \\ \mathcal{C}_2(q) &= \mathcal{C}_1(m_W)\left(\frac{3\alpha_s(q)}{4\pi}\log\frac{q^2}{m_W^2}\right)\,, \end{aligned} \tag{1.25}$$

which indeed reproduces the logarithmic terms.

1.5.3 Concluding remarks

Let us summarize what we have learned in a sentences without getting lost in lengthy computations.

Resummation: We have explicitely seen that multi-scale loop-amplitudes will generically contain logarithms of ratios of the various scales. For strong scale hierarchies, these logarithms can be problematic for the convergence of our perturbation expansion. Instead of counting them as suppressed by the coupling α, we count them as inverse powers of the coupling such the combination of both is of the same order as the leading term. Schematically, we rearrange our series as follows:

$$\begin{aligned} &1+\frac{\alpha}{4\pi}\left(a_1\log\lambda+c_1\right)+\frac{\alpha^2}{16\pi^2}\left(a_2\log^2\lambda+b_2\log\lambda+c_2\right)+\dots \\ &=\left(1+a_1\frac{\alpha}{4\pi}\log\lambda+a_2\frac{\alpha^2}{16\pi^2}\log^2\lambda\right)+\left(\frac{\alpha}{4\pi}c_1+b_2\frac{\alpha^2}{16\pi^2}\log\lambda\right)+\frac{\alpha^2}{16\pi^2}c_2\,. \end{aligned} \tag{1.26}$$

In the first line, terms are sorted by their order in α, whereas in the second line, they are ordered by our new counting scheme. The two schemes are called *fixed-order perturbation theory* (first line) and *RG-improved perturbation theory* (second line). The different orders in the fixed-order scheme are referred to as leading order (LO), next-to-leading order (NLO), et cetera, for fixed-order perturbation theory. When computations are done in RG-improved perturbation theory, one refers to the different orders as leading log (LL), next-to-leading log (NLL), et cetera. Thus when results are being quoted as a certain order in logarithmic counting, it is done so to indicate that logarithms have been resummed.

Using the RG we have shown that we can indeed obtain the logarithmic terms to all orders in perturbation theory: We have removed one of the scales by splitting the theory into several theories that each only contain a single scale. Matching and the method of region immediately shows how the full result *factorizes* into a product of something that contains only the hard scale (the Wilson coefficients) and something that contains only the soft scale (the EFT matrix element). Since we are free to do this splitting at any scale, we can choose one at which the logarithms are small or even vanish so the result contains all powers of the logarithms to a good approximation. We then use the renormalization group to move this scale to the physical value. Since the RG evolution is the solution to imposing independence of this scale, we get a result at a different scale that still contains all powers of these logarithms, even if they are no longer small.

There is a range of effective theory constructions that are completely driven by resummation in the sense that some result contains large logarithms, that need to be resummed. Then one identifies the relevant physics contributing the different scales and build an EFT that separates them. By renormalizing the theory, one resums the large logarithms. Many factorization theorems, especially in QCD, can be derived this way.

Renormalization: The fact that the theory continuously changes with the variation of the scale μ goes back to our original method of splitting the theory into regions, $\phi = \phi_H + \phi_S$. By lowering the scale, we continuously shuffle Fourier modes of the fields from ϕ_S to ϕ_H, even though we might not cross thresholds of heavy particles being integrated out. With lower and lower scales, the remaining modes of the theory are of longer and longer wavelengths. The shortest distance these fields can resolve thus becomes longer and longer. Physics at shorter distances then appear as local effects. This is reflected in the running of the short-distance coefficients, the couplings of the theory.

The above sentiment is not limited to EFTs, it holds for the renormalization group in general. It is then safe to say that every renormalized theory is an effective theory. We can even interpret counterterms in a way less ad-hoc than they appeared before: We know that “physics is finite”. When our loop computations give divergent result, we are simply extrapolating our theory into a region in which it probes short-distance effects that we are approximating to be local. In the full theory, some new effect - we do not know what - becomes relevant at high scales, preventing the amplitude from diverging. If these new high-scale effects are integrated out, they appear as an effective interaction cancelling the divergence in the effective theory, just as our Wilson coefficients cancel the divergences of the EFT matrix elements.