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Applications of General Relativity

Solutions 1

Spring 2017
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Exercise 1 [The cosmological constant in the black hole context]

a) In vacuum, $T_{\mu\nu} = 0$. Taking the trace of the field equations we get

$$R = -4\Lambda. \tag{1}$$

Substituting this back in to the field equations gives

$$R_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{2}$$

b) We follow the same procedure here as we did for the Schwarzschild solution. From expressions for the components of the Ricci tensor for a Schwarzschild ansatz, (see the notes from the previous semester, or Weinberg), we have that

$$\frac{R_{rr}}{A} + \frac{R_{tt}}{B} = -\frac{1}{rA}\left(\frac{A'}{A} + \frac{B'}{B}\right). \tag{3}$$

This leads to

$$\frac{d}{dr}\ln(AB) = 0 \Longrightarrow AB = const. = 1. \tag{4}$$

Considering our third equation, $R_{\theta\theta} + \Lambda g_{\theta\theta} = 0$, we have

$$R_{\theta\theta} + \Lambda g_{\theta\theta} = 1 + \frac{r}{2A}\left(\frac{A'}{A} - \frac{B'}{B}\right) - \frac{1}{A} - \Lambda r^2 = 0, \tag{5}$$

which simplifies to a first-order linear differential equation for $B(r)$,

$$1 - rB' - B - \Lambda r^2 = 0, \tag{6}$$

which is easily solved by the ansatz $B(r) = 1 - \frac{2GM}{r} + B_\Lambda(r)$. The result is

$$B_\Lambda(r) = -\frac{\Lambda}{3}r^2. \tag{7}$$

c) As discussed in GR I, the bending angle can be written as

$$\Delta\phi = 2|\phi(r_0) - \phi_\infty| - \pi, \tag{8}$$

where r_0 is the distance of closest approach. For a photon in a static isotropic metric we have the formula (GR I 23.19)

$$\phi(r) - \phi_\infty = \int_r^\infty \frac{\sqrt{B^{-1}}}{\sqrt{\left(\frac{r}{r_0}\right)^2 \frac{B(r_0)}{B(r)} - 1}}\frac{dr}{r} \tag{9}$$

$$= \int_r^\infty \sqrt{\frac{r_0^2}{r^2 B(r_0) - r_0^2 B(r)}}\frac{dr}{r}. \tag{10}$$

It turns out that, because the term $B_\Lambda(r)$ is $\propto r^2$, the terms exactly cancel each other. This means that light deflection does not depend on Λ, and therefore we cannot use the deflection of light as a tool to determine Λ, an important observational consequence.

Exercise 2 [Age of the universe]

From the lecture you know that the Friedmann equation for a flat Universe can be written as

$$H(t)^2 = \left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho_\mathrm{m}(t) + \frac{\Lambda}{3}. \tag{11}$$

Using that the matter density scales as $\rho_\mathrm{m}(t) = \rho_{\mathrm{m},0}a(t)^{-3}$ and the definition of the present day density parameters

$$\Omega_{\mathrm{m},0} = \frac{8\pi G}{3H_0^2}\rho_{\mathrm{m},0} \qquad \Omega_{\Lambda,0} = \frac{\Lambda}{3H_0^2} \tag{12}$$

one can easily derive

$$H(t)^2 = H_0^2\left(\Omega_{\mathrm{m},0}a(t)^{-3} + \Omega_{\Lambda,0}\right). \tag{13}$$

The age of the Universe is now given by

$$t = \int_0^{t_0} \mathrm{d}t = \int_0^1 \frac{\mathrm{d}a}{\dot{a}} = \int_0^1 \frac{\mathrm{d}a}{a\,H(a)} = 1.3870 \times 10^{10}\,\mathrm{y}, \tag{14}$$

where we used that that $1/H_0 = 1.4386 \times 10^{10}\,\mathrm{y}$ and evaluated the integral numerically.
The comoving distance traveled by a photon on a radial null geodesic is given by ($\mathrm{d}s = 0 \Rightarrow c\,\mathrm{d}t = a\,\mathrm{d}r$)

$$r = c\int_0^{t_0} \frac{\mathrm{d}t}{a} = c\int_0^1 \frac{\mathrm{d}a}{a\dot{a}} = c\int_0^1 \frac{\mathrm{d}a}{a^2\,H(a)} = 14\,571\,\mathrm{Mpc}. \tag{15}$$

Note that the Hubble length $c/H_0 \approx 4409\,\mathrm{Mpc}$ is a typical length scale for causality in our expanding Universe.