MMP I
Tutorial 12

HS 2019
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Exercise 1: Integral equations (5 Pts.)
Consider the integral operator

$$|x\rangle \xrightarrow{T} T|x\rangle \sim (Tx)(t) = \int_0^1 \mathrm{d}s\Big(1 - 3st\Big)x(s) \tag{1.1}$$

in the Hilbert space $L^2[0,1]$ and $|x\rangle \in L^2[0,1]$. Note that the suitably normalised Legendre polynomials provide an orthonormal basis of $L^2[0,1]$.

a) Determine all eigenvalues λ_i and all eigenvectors $|\phi_i\rangle \sim \phi_i(t)$ of the operator T.

b) Consider the following equation:

$$(I - \lambda T)|x\rangle = |y\rangle \tag{1.2}$$

where $|y\rangle \sim y(t) = 3t^2$. Find the general solution of the equation depending on the value of λ. For which values of λ does a solution exist?

Exercise 2: Laplace operator in non-cartesian coordinates (4 Pts.)

a) Derive the Laplace operator in cylindrical coordinates r, ϕ, z, defined through $x = r\cos\phi$, $y = r\sin\phi$, $z = z$.

b) Show the equivalence of the following three differential operators:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}, \qquad \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}, \qquad \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}. \tag{2.1}$$

Exercise 3: Laplace operator on a circle (4 Pts.)
Consider the two-dimensional Laplace equation $\nabla^2\psi(r,\phi) = 0$ and let C_R be the circle of radius R centred at the origin.

a) Find the solution to $\nabla^2\psi = 0$ that is regular inside C_R and satisfies the Dirichlet boundary condition $\psi(R,\phi) = \sin^2(2\phi)$ on C_R.

b) Find the solution to $\nabla^2\phi = 0$ that is regular outside C_R and satisfies the Dirichlet boundary condition $\psi(R,\phi) = \sin^2(2\phi)$ on C_R.