# CALCULUS OF VARIATIONS



We consider the integral

$$\mathcal{F}(y) = \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx$$

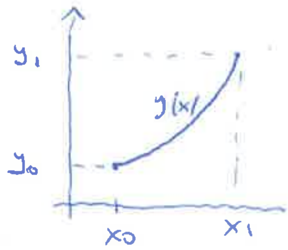
$F$ given function of class $C^2$

$\mathcal{F} : y \to \mathbb{R}$ is called functional

$y(x_0) = y_0 \qquad y(x_1) = y_1$

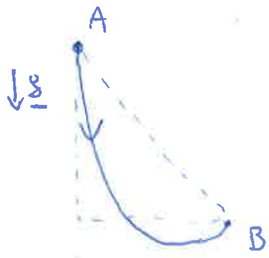
PROBLEM: find the function $y : [x_0, x_1] \to \mathbb{R}$ such that the integral $\mathcal{F}(y)$ is MINIMUM

Although $(x_0, y_0)$ and $(x_1, y_1)$ are given the actual trajectory in the $(x, y)$ plane is unknown




Typical physical problems where the calculus of variations is relevant:

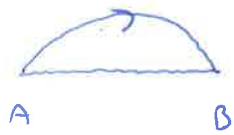
- Brachistochrone curve
  ↳ greek: shortest time




find the curve for which the minimum falling time is obtained given two points A and B

Contrary to what one could expect the minimum time is not obtained on a straight line but on a cycloid

- Find the trajectory of an airplane for which the fuel consumption is minimum




- Minimal action principle in classical mechanics

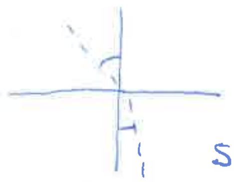
  classical trajectory → minimal action

$$S = \int dt\, L(x, \dot{x})$$

↳ action functional

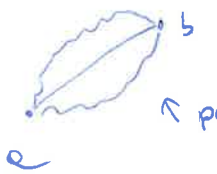
- Fermat principle

The path followed by a ray of light is the one that can be traversed in the shortest time



Snell law (see later)

- Quantum mechanics




↑ possible trajectories

$$\mathcal{M}(a \to b) = \sum_{P} e^{iS(x)/\hbar}$$

↑ sum over the possible paths

↑ action evaluated over the path

The amplitude to go from a to b is obtained by summing the contribution from each path. In the classical limit we have $S(x) \gg \hbar$ $\Rightarrow$ since $S$ is stationary over the classical path all the other contributions cancel out due to the rapidly oscillating phase factor and only the classical one survives.

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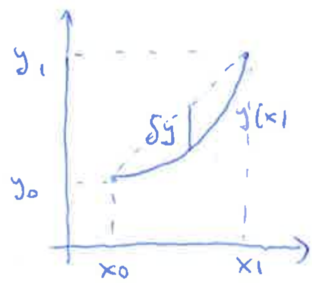
We now go back to our problem and look for the conditions to make the functional $J(y)$ minimum.

$$J(y) = \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx$$




We can parametrize the difference between the actual path and the optimal path through a parameter $\alpha$

$$y(x,\alpha) = y(x,0) + \alpha\,\eta(x) \qquad \text{with } \eta(x_0) = \eta(x_1) = 0$$

$$\delta y = \frac{\partial y}{\partial \alpha}$$

A necessary conditions such that the functional is minimum is that $\delta J = \dfrac{dJ}{d\alpha} = 0$

$$\frac{dJ}{d\alpha} = \int_{x_0}^{x_1} \left( \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) dx = 0$$

$$\delta y' = \frac{\partial y'}{\partial \alpha} = \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial \alpha} \right) = (\delta y)'$$

$$\Rightarrow \frac{\partial F}{\partial y'} \delta y' = \frac{\partial F}{\partial y'} (\delta y)' = \left( \frac{\partial F}{\partial y'} \delta y \right)' - \left( \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y$$

$$\Rightarrow 0 = \int_{x_0}^{x_1} \left( \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y + \cancel{\frac{\partial F}{\partial y'} \delta y \Big|_{x_0}^{x_1}}$$

↳ vanishes because $\delta y(x_0) = \delta y(x_1) = 0$

But $\delta y = \eta(x)$ is arbitrary, except for the conditions $\eta(x_0) = \eta(x_1) = 0$

$$\Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \qquad \text{EULER EQUATION}$$

## SPECIAL CASES

- Let us suppose that $\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \Rightarrow \frac{\partial F}{\partial y'} = \text{const}$

$\frac{\partial F}{\partial y'}$ is a "first integral of motion"

- We now suppose that $\frac{\partial F}{\partial x} = 0$, that is, F is a function of y and y' only

The Euler equation is

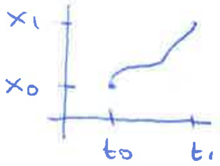
$$\left( \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) y' = 0 \Rightarrow \frac{dF}{dx} - \frac{\partial F}{\partial y'} y'' - \left( \frac{d}{dx} \frac{\partial F}{\partial y'} \right) y' = 0$$

$$= \frac{d}{dx} \left( F - \frac{\partial F}{\partial y'} y' \right)$$

## example

$$S = \int_{t_0}^{t_1} L(x, \dot{x})\, dt$$ action

$L(x, \dot{x})$ Lagrangian that does not explicitly depend on time




$x = x(t)$ trajectory

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{\partial L}{\partial \dot{x}} \dot{x} - L = \text{const}$$ "first integral of motion"

$\rightarrow$ Hamiltonian

$\rightarrow$ energy is conserved

- $$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$$

$$\Rightarrow F = F(y') \qquad \frac{d}{dx}\frac{\partial F}{\partial y'} = 0 = \frac{\partial^2 F}{\partial y'^2} y'' \Rightarrow \text{if } \frac{\partial^2 F}{\partial y'^2} \neq 0$$

we must have $y'' = 0$

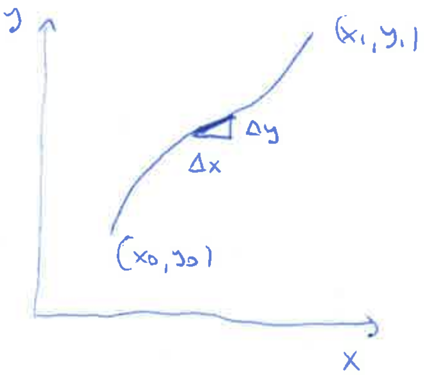
$\Rightarrow$ y is linear

## EXAMPLE

Find the shortest path connecting two points $(x_0, y_0)$ and $(x_1, y_1)$

$$\mathfrak{Z}(y) = \int_{P_0}^{P_1} ds = \int_{P_0}^{P_1} (dx^2 + dy^2)^{1/2} = \int_{x_0}^{x_1} \sqrt{1 + y'^2}\, dx$$

line integral



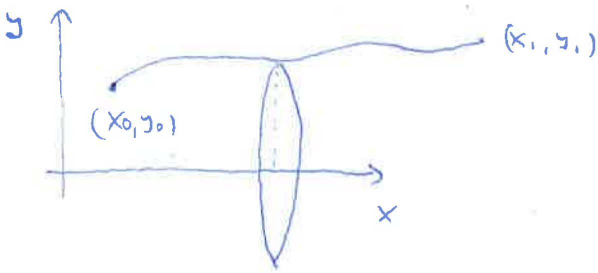

The line element is $\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$ $\qquad F = \sqrt{1 + y'^2}$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial F}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}} = \text{const} \qquad y'^2 = A^2(1 + y'^2) \qquad y' = \text{const}$$

$\Rightarrow$ the solution is a straight line connecting $P_0$ and $P_1$

## Example

Given two points $(x_0, y_0)$, $(x_1, y_1)$ find the curve such that the surface of the generated solid is minimum.




$$dA = 2\pi y \, ds = 2\pi y \sqrt{dx^2 + dy^2} \quad \Rightarrow \quad J(y) = \int_{x_0}^{x_1} 2\pi \, y(x) \sqrt{1+y'^2} \, dx$$

$$F(x, y, y') = y\sqrt{1+y'^2} \qquad \text{since } \frac{\partial F}{\partial x} = 0 \quad \Rightarrow \quad F - \frac{\partial F}{\partial y'} y' = \text{const}$$

$$y\sqrt{1+y'^2} - \frac{y y'^2}{\sqrt{1+y'^2}} = \text{const} \qquad \frac{y}{\sqrt{1+y'^2}} = c \qquad \frac{y^2}{1+y'^2} = c^2$$

$$1 = \frac{y^2}{c^2} - y'^2 \qquad \text{remember that} \quad \cos^2 hx - \sin^2 hx = 1 \qquad \frac{d}{dx} \cosh x = \sinh x$$

$$\Rightarrow \quad y = c \cosh\left(\frac{x}{c} - d\right)$$ use boundary conditions to fix $c$ and $d$

catenoid

## Natural boundary conditions

Up to now we have been considering the case $y(x_0) = y_0$ and $y(x_1) = y_1$.

What happens if $y(x_1)$ is left arbitrary?

In the derivation of the Euler equation we had

$$0 = \int_{x_0}^{x_1} \left( \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y + \left. \frac{\partial F}{\partial y'} \delta y \right|_{x_0}^{x_1}$$

Now, if $\delta y(x_1)$ is arbitrary we need an additional condition that $\left. \frac{\partial F}{\partial y'} \right|_{x = x_1} = 0$

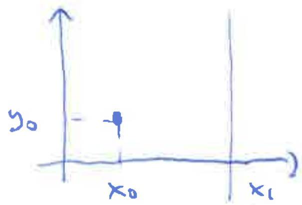
We can also consider the case in which $\delta y(x_0)$ is arbitrary and $\delta y(x_1) = 0$.

We can even consider the case in which both $\delta y(x_0)$ and $\delta y(x_1)$ are arbitrary.

The necessary boundary conditions are called natural boundary conditions.

EXAMPLE

Shortest distance from $(x_0, y_0)$ to a straight line at $x = x_1$




$$J(y) = \int_{x_0}^{x_1} \sqrt{1+y'^2}\, dx$$

Euler equation: $y$ is linear

$y = ax + b \qquad y_0 = a x_0 + b \qquad \Rightarrow$ we need another condition!

$$\left.\frac{\partial F}{\partial y'}\right|_{x=x_1} = 0 = \left.\frac{y'}{\sqrt{1+y'^2}}\right|_{x_1} \Rightarrow y'(x_1) = 0 \Rightarrow y = y_0 \text{ is the solution!}$$

—

We now ask ourselves the following question: can we modify the function $F$ such that the Euler equation remains the same?

$$F(x,y,y') \to F(x,y,y') + \frac{dG(x,y)}{dx}$$ where $G(x,y)$ does not depend on $y'$

$$\delta \int_{x_0}^{x_1} \left(F + \frac{dG}{dx}\right) dx = \int_{x_0}^{x_1} \left[ \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \frac{\partial}{\partial y}\left(\frac{dG}{dx}\right)\delta y + \frac{\partial}{\partial y'}\left(\frac{dG}{dx}\right)\delta y' \right]$$

$$= \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right)\delta y + \left.\frac{\partial F}{\partial y'}\delta y\right|_{x_0}^{x_1}$$

$$+ \int_{x_0}^{x_1} \frac{\partial}{\partial y}\left(\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} y'\right)\delta y + \int_{x_0}^{x_1} \frac{\partial}{\partial y'}\left(\frac{\partial G}{\partial x} + \frac{\partial G}{\partial y} y'\right)\delta y'$$

$$= \int_{x_0}^{x_1} \left( \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y + \frac{\partial F}{\partial y'} \delta y \Big|_{x_0}^{x_1}$$

$$+ \int_{x_0}^{x_1} \left( \frac{\partial^2 G}{\partial x \partial y} + \frac{\partial^2 G}{\partial y^2} y' \right) \delta y + \int_{x_0}^{x_1} \frac{\partial G}{\partial y} \delta y'$$

$$= \int_{x_0}^{x_1} \left( \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y + \frac{\partial F}{\partial y'} \delta y \Big|_{x_0}^{x_1}$$

$$+ \int_{x_0}^{x_1} \left( \cancel{\frac{\partial^2 G}{\partial x \partial y}} + \cancel{\frac{\partial^2 G}{\partial y^2} y'} - \cancel{\frac{\partial^2 G}{\partial x \partial y}} - \cancel{\frac{\partial^2 G}{\partial y^2} y'} \right) \delta y + \frac{\partial G}{\partial y} \delta y \Big|_{x_0}^{x_1}$$

$\Rightarrow$ in case of fixed boundary conditions ( $\delta y(x_0) = \delta y(x_1) = 0$ ) the Euler equation is unchanged. In case of natural boundary conditions (say $\delta y(x_1)$ arbitrary) the natural boundary condition is modified as

$$\frac{\partial F}{\partial y'}\Big|_{x_1} = 0 \longrightarrow \frac{\partial F}{\partial y'}\Big|_{x_1} + \frac{\partial G}{\partial y}\Big|_{x_1} = 0$$

What happens if the function F <u>depends on higher derivatives</u> ?

Suppose $F = F(x, y, y', y'')$

$\delta y' = (\delta y)'$ $\quad \delta y'' = (\delta y)''$

$$\delta J = \int_{x_0}^{x_1} \left( \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' + \frac{\partial F}{\partial y''} \delta y'' \right) dx$$

$$= \int_{x_0}^{x_1} \left( \frac{\partial F}{\partial y} \delta y - \left( \frac{d}{dx} \frac{\partial F}{\partial y'} \right) \delta y - \frac{d}{dx} \frac{\partial F}{\partial y''} \delta y' \right) dx$$

$$+ \frac{\partial F}{\partial y'} \delta y \Big|_{x_0}^{x_1} + \frac{\partial F}{\partial y''} \delta y' \Big|_{x_0}^{x_1}$$

leads to modified Euler eq.

$$= \int_{x_0}^{x_1} \left( \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial F}{\partial y''} \right) \delta y + \frac{\partial F}{\partial y'} \delta y \Big|_{x_0}^{x_1} + \frac{\partial F}{\partial y''} \delta y' \Big|_{x_0}^{x_1} - \frac{d}{dx} \frac{\partial F}{\partial y''} \delta y \Big|_{x_0}^{x_1}$$

$\Rightarrow$ We conclude that when $F$ depends on higher derivatives the Euler equation is modified and also that we need boundary conditions not only on $y(x_0)$ and $y(x_1)$ but also on $y'(x_0)$ and $y'(x_1)$ (or higher).

In the case in which $F = F(x, y, y', y'')$ we have $\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial F}{\partial y''} = 0$

+

What happens when there is more than one independent variable?

$u = u(x,y)$ $\qquad$ $J(u) = \int_G F(x, y, u, u_x, u_y)\, dx\, dy$

We can provide the boundary conditions by fixing $u(x,y)$ on the boundary of $G$.

In this case we have

$$\delta J = \int_G \left( \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u_x}\delta u_x + \frac{\partial F}{\partial u_y}\delta u_y \right) dx\, dy$$

$\Rightarrow$ The Euler equation becomes

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\frac{\partial F}{\partial u_x} - \frac{d}{dy}\frac{\partial F}{\partial u_y} = 0$$

$$\frac{d}{dx} = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx}\frac{\partial}{\partial u_x}$$

$$\frac{d}{dy} = \frac{\partial}{\partial y} + u_y \frac{\partial}{\partial u} + u_{yy}\frac{\partial}{\partial u_y}$$

EXAMPLE

Energy density of the electric field $\quad F = \frac{1}{2}\varepsilon \underline{E}^2 \qquad \underline{E} = -\nabla\varphi$

$$= \frac{1}{2}\varepsilon(\nabla\varphi)^2$$

$$J = \int (\nabla\varphi)^2\, dx\, dy\, dz = \int \left( \left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 + \left(\frac{\partial\varphi}{\partial z}\right)^2 \right) dx\, dy\, dz$$

$\Rightarrow$ The Euler equation is $\quad -2(\varphi_{xx} + \varphi_{yy} + \varphi_{zz}) = 0 \quad \Rightarrow \quad \nabla^2\varphi = 0 \quad$ Laplace equation;

→ What happens if there are more unknown functions?

$$\mathcal{J}(y,z) = \int_{x_0}^{x_1} F(x, y(x), z(x), y'(x), z'(x))\, dx$$

$y(x_0) = y_0 \qquad z(x_0) = z_0$

$y(x_1) = y_1 \qquad z(x_1) = z_1$

$$y \to y + \alpha\,\delta y \quad \Rightarrow \quad \frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

$$z \to z + \alpha\,\delta z \quad \Rightarrow \quad \frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} = 0$$

two unknown functions

→ two Euler eqs.

APPLICATIONS : CLASSICAL MECHANICS

Let us consider a classical system characterized by the degrees of freedom $q_1 \cdots q_M$

The time is now our independent variable and $F \to L(q_1, q_2 \cdots q_M, \dot{q}_1 \cdots \dot{q}_M) \equiv L(q_i, \dot{q}_i)$

$$S = \int_{t_0}^{t_1} L(q_i, \dot{q}_i)\, dt \qquad \text{ACTION}$$

↳ Lagrangian of the system

↳ the classical "trajectory" (motion) of the system is the one that makes the action stationary

Since there are M independent functions ⇒ M Euler equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0 \qquad 1 \leq i \leq M$$

exemple

Motion of one particle of mass $m$ in a potential $U(x,y,z,t)$

$$L = T - U = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(x,y,z,t)$$

Euler eqs : $\begin{cases} -U_x = m\ddot{x} \\ -U_y = m\ddot{y} \\ -U_z = m\ddot{z} \end{cases}$ ⇒ this is indeed what we get from Newton's law!

Note that if $U$ does not depend on $t$ we can write:

$$0 = \sum_i \left( \frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} \right) \dot{x}_i$$

vanishes because of Euler eqs

$x_1 = x$
$x_2 = y$
$x_3 = z$

$$= \frac{dL}{dt} - \sum_i \frac{\partial L}{\partial \dot{x}_i} \ddot{x}_i - \sum_i \left( \frac{d}{dt} \frac{\partial L}{\partial \dot{x}_i} \right) \dot{x}_i$$

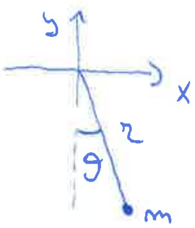
$$= \frac{d}{dt} \left( L - \sum_i \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i \right) \Rightarrow -L + \sum_i \frac{\partial L}{\partial \dot{x}_i} \dot{x}_i = \text{constant}$$

$= H$ Hamiltonian!

$$H = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + U(x, y, z)$$

it is the energy of the system

<u>example</u> : pendulum




we look for a function $\vartheta(t)$ such that $\int L(\vartheta, \dot{\vartheta}, t)\, dt$ is stationary

$x = r \sin\vartheta \qquad y = -r\cos\vartheta$

$L = T - U$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \left( (r \cos\vartheta\, \dot{\vartheta})^2 + (r \sin\vartheta\, \dot{\vartheta})^2 \right)$$

$$= \frac{1}{2} m r^2 \dot{\vartheta}^2$$

$U = -mgr\cos\vartheta$

$$\Rightarrow L = \frac{1}{2} m r^2 \dot{\vartheta}^2 + mgr\cos\vartheta$$

$$\frac{\partial L}{\partial \vartheta} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\vartheta}} = 0 \qquad -\cancel{m}gr\sin\vartheta - \frac{d}{dt} \cancel{m} r^2 \dot{\vartheta} = 0 \qquad \ddot{\vartheta} + \frac{g}{r} \sin\vartheta = 0$$

Two boundary conditions are required ( e.g. $\vartheta(t_0)$ and $\dot{\vartheta}(t_0)$)

## APPLICATIONS: GEOMETRICAL OPTICS

The equations of Geom optics are the Euler equations corresponding to the <u>Fermat principle</u>: a light ray from two points $P_0$ and $P_1$ follows the path that makes the time $T = \int_{P_0}^{P_1} dt$ minimum.

Defining $c = v\,n$ where $c$ is the light velocity in vacuum, $v$ is the light velocity in the medium and $n$ the <u>refractive index</u> we can write

$$T = \int_{P_0}^{P_1} dt = \int_{P_0}^{P_1} \frac{ds}{v} = \frac{1}{c}\int_{P_0}^{P_1} n\, ds \qquad ds = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\; dt$$

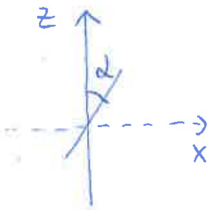
$\Rightarrow$ We have to minimize

$$\int_{t_0}^{t_1} \underbrace{n(x(t),y(t),z(t))\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}_{F(t,x,y,z,\dot{x},\dot{y},\dot{z})}\; dt$$

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial \dot{x}} = 0 = \frac{\partial n}{\partial x}\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2} - \frac{d}{dt}\left[ n \frac{\dot{x}}{\sqrt{\phantom{x}}} \right]$$

$$\Rightarrow \quad \nabla n = \frac{d}{ds}\left( n \frac{d\underline{x}}{ds} \right)$$

Special case: $n = n(z)$ $\qquad x = s \sin\alpha$, $y = 0$




$\Rightarrow \quad n \sin\alpha = \text{const}$

$\Rightarrow$ If we have a boundary at $z=0$ such that $n = n_1$ for $z<0$ and $n = n_2$ for $z>0$ we have $\quad n_1 \sin\alpha_1 = n_2 \sin\alpha_2 \quad$ <u>SNELL LAW</u>

# Variation calculus

## with constraints

Suppose that we have a variational problem $J(y) = \int_{x_0}^{x_1} F(x,y,y')\,dx$ subject on a constraint on $K(y) = \int_{x_0}^{x_1} G(x,y,y')\,dx = C$ (fixed constant)

The problem can be solved by considering the variational problem

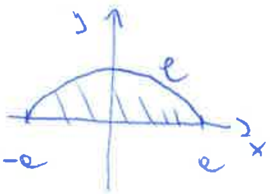
$$J - \lambda K = \int_{x_0}^{x_1} \left(F(x,y,y') - \lambda G(x,y,y')\right) dx$$

$\lambda$ Lagrange multiplier *

$\Rightarrow$ the Euler eq. reads

$$(F - \lambda G)_y - \frac{d}{dx}(F - \lambda G)_{y'} = 0$$

## Example




What is the curve with fixed length $\ell$ which gives the largest surface?

$$F(y) = \int_{-\ell}^{\ell} y\,dx \qquad K(y) = \int_{-\ell}^{\ell} \sqrt{1+y'^2}\,dx = \ell$$

The Euler equation gives

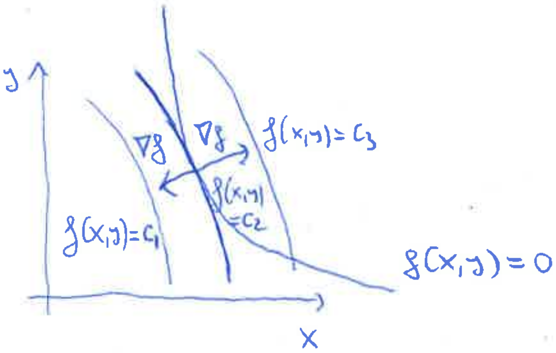
$$1 - \lambda\left(-\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}\right) = 0 \qquad \lambda \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = -1$$

$$\lambda \frac{y'}{\sqrt{1+y'^2}} = k_1 - x$$

$$\Rightarrow \lambda^2 \frac{y'^2}{1+y'^2} = (k_1 - x)^2 \qquad \lambda^2 y'^2 = (1+y'^2)(k_1-x)^2$$

$$\Rightarrow y' = \frac{k_1 - x}{\sqrt{\lambda^2 - (k_1-x)^2}} \Rightarrow y = \sqrt{\lambda^2 - (k_1 - x)^2} + k_2$$ CIRCUMFERENCE!

(*) Lagrange multipliers




↑

level curve of $f(x,y)$ and $g(x,y)=0$

Find the extrema of $f(x,y)$ under the condition $g(x,y)=0$

Suppose we walk along the line $g(x,y)=0$ and we look for the points on which $f$ does not change

$\Rightarrow$ the curve $g(x,y)=0$ must be tangent to a <u>level curve</u> of $f(x,y)$

$\Rightarrow$ Since the gradient is ortogonal to the level curves we must have that <u>the gradients are parallel</u> $\nabla f = \lambda \nabla g$

This means that we can study the function $f(x,y) - \lambda\, g(x,y)$