Taylor - Couette

Navier-Stokes-Eq

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} = \nu \vec{\nabla}^2 \vec{v} - \vec{\nabla} p/\varrho + g\vec{e}_x$$

$\downarrow$

stationary flow: $\frac{\partial}{\partial t} = 0$ $\rightarrow$ inviscid flow. $\rightarrow$ incompressible flow $\vec{\nabla}^2 \vec{v} = 0$

put this into cylindrical coordinates: $\frac{\partial}{\partial \varphi} = 0$ $\frac{\partial}{\partial z} = 0$ $v_z = 0$ $v_r = 0$

v_φ ; $\frac{\partial}{\partial r} \neq 0$ $\quad$ ↳ for stationary shear-flow

$\Rightarrow \vec{\nabla}^2 \vec{v} = 0$ will correspond to $\boxed{\frac{\partial^2 v_\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\varphi}{\partial r} - \frac{v_\varphi}{r^2} = 0}$

$v_\varphi = r\omega$ $\qquad v_\varphi(r) \rightarrow \omega(r)$

Ansatz: $\boxed{v_\varphi(r) = ar + \frac{b}{r}}$ $\qquad$ a and b given by boundary conditions

$$\frac{\partial v_\varphi}{\partial r} = a - \frac{b}{r^2} \qquad \frac{\partial^2 v_\varphi}{\partial r^2} = + \frac{2b}{r^3}$$

$$\Rightarrow \frac{2b}{r^3} + \frac{1}{r}\left(a - \frac{b}{r^2}\right) - \frac{\left(ar + b/r\right)}{r^2} \overset{!}{=} 0$$

$$\cancel{\frac{2b}{r^3}} + \cancel{\frac{a}{r}} - \cancel{\frac{b}{r^3}} - \cancel{\frac{a}{r}} - \cancel{\frac{b}{r^3}} = 0$$

r_1 = Radius of inner cylinder

Ω_1 = angular freq. of inner cylinder

r_2 = Radius of outer cylinder

$V_\varphi(r_1) = \Omega_1 r$

$V_\varphi(r_1) = \Omega_1 r_1$

$V_\varphi(r_2) = 0 \qquad (= \Omega_2 r_2)$

general solution

$V_\varphi(r_1) = a r_1 + \frac{b}{r_1} = \Omega_1 r_1 \qquad V_\varphi(r_2) = a r_2 + \frac{b}{r_2} = \Omega_2 r_2$

$\hookrightarrow b = r_2^2(\Omega_2 - a)$

$a r_1^2 + \frac{r_2^2(\Omega_2 - a)}{} = \Omega_1 r_1^2$

$a(r_1^2 - r_2^2) = \Omega_1 r_1^2 - \Omega_2 r_2^2 \Rightarrow \boxed{a = \frac{\Omega_1 r_1^2 - \Omega_2 r_2^2}{r_1^2 - r_2^2}}$

$b = r_2^2\left(\Omega_2 - \frac{\Omega_1 r_1^2 - \Omega_2 r_1^2}{r_1^2 - r_2^2}\right)$

$\boxed{b = \frac{r_2^2 r_1^2(\Omega_2 - \Omega_1)}{r_1^2 - r_2^2}}$

non-rotating outer cylinder: $\Omega_2 = 0$

$\boxed{a = \Omega_1 \frac{r_1^2}{r_1^2 - r_2^2} \qquad b = \frac{r_2^2 r_1^2}{r_2^2 - r_1^2}\Omega_1}$

Stability condition:

$2 r \omega^2 + r^2 \omega \frac{d\omega}{dr} > 0 \longleftarrow \omega = a + \frac{b}{r^2} \qquad \frac{d\omega}{dr} = -\frac{2b}{r^3}$

$\cancel{2} r^2 \left(a + \frac{b}{r^2}\right)^{\cancel{2}} = r^2 \cancel{\left(a + \frac{b}{r^2}\right)} \frac{\cancel{2}b}{\cancel{r^3}} \qquad$ @ instability

$$r^2\left(a + \frac{b}{r^2}\right) = 1 \quad \Rightarrow \quad a r^2 = -b \qquad \text{instability condition}$$

Scaling the Navier-Stokes eq.

$$\vec{r} = \vec{R} \cdot l \qquad \vec{v} = \vec{u}\langle \vec{v} \rangle \qquad t = \tau \cdot \frac{l}{\langle v \rangle}$$

$$\frac{\langle v \rangle^2}{l} \frac{\partial \vec{u}}{\partial \tau} + \frac{\langle v \rangle^2}{l} \vec{u} \cdot \vec{\nabla}_R \vec{u} = + \frac{\nu}{l^2} \vec{\nabla}_R^2 \vec{u} \, \langle v \rangle$$

$$\frac{l \langle v \rangle^{\cancel{2}}}{\nu \cancel{\langle v \rangle}} \left(\frac{\partial \vec{u}}{\partial \tau} + \vec{u} \, \vec{\nabla}_R \vec{u} \right) = \frac{\cancel{\langle v \rangle}}{\cancel{l^2}} \vec{\nabla}_R^2 \vec{u}$$

$$\underbrace{\frac{l \langle v \rangle}{\nu}}_{Re} \left(\quad \right) = \vec{\nabla}_R^2 \vec{u}$$