MMP I
Solution Sheet 7

HS 21
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Exercise 1 [Ordinary differential equations (4 points)]

a) $y' = \frac{\sin^2(x) - \sin(x)\sin(y)}{\cos^2(y) - \cos(x)\cos(y)} = \frac{dy}{dx}$, Is it an exact differential equation?

$$\underbrace{(\cos^2(y) - \cos(x)\cos(y))}_{Q} dy = \underbrace{(\sin^2(x) - \sin(x)\sin(y))}_{-P} dx$$

$$\underbrace{Pdx + Qdy}_{df \Rightarrow f = const} = 0$$

$$\frac{\partial P}{\partial y} = \sin(x)\cos(y)$$

$$\frac{\partial Q}{\partial x} = \sin(x)\cos(y)$$

$\Rightarrow$ The equation is exact, $P = \frac{df(x,y)}{dx}$; $Q = \frac{df(x,y)}{dy}$

$$\begin{aligned}
f(x,y) &= \int P dx \\
&= \int \sin(x)\sin(y)dx - \int \sin^2(x)dx \\
&= -\cos(x)\sin(y) - \frac{1}{2}(x - \sin(x)\cos(x)) + C(y) \\
\frac{\partial f}{\partial y} &= -\cos(x)\cos(y) + C'(y) = Q = -\cos(x)\cos(y) + \cos^2(y) \\
C'(y) &= \cos^2(y) \\
C(y) &= \int \cos^2(y)dy = \frac{1}{2}(\sin(y)\cos(y) + y) \\
f(x,y) &= -\cos(x)\sin(y) - \frac{1}{2}(x - \sin(x)\cos(x)) + \frac{1}{2}(\sin(y)\cos(y) + y)
\end{aligned}$$

The solution to the differential equation is $f(x,y) = const$

$-\cos(x)\sin(y) - \frac{1}{2}(x - \sin(x)\cos(x)) + \frac{1}{2}(\sin(y)\cos(y) + y) = const$

b) $\vec{F} = q\vec{v} \times \vec{B} \rightarrow m\ddot{\vec{r}} = q\dot{\vec{r}} \times \vec{B}$

Let's choose the z-axis $||$ $\vec{B}$, so

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \vec{v} = \dot{\vec{r}} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \ddot{\vec{r}} = \begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{pmatrix} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix}, \vec{B} = \begin{pmatrix} 0 \\ 0 \\ B_z \end{pmatrix}$$

$$\rightarrow \quad m \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = q \begin{pmatrix} B_z\dot{y} \\ -B_z\dot{x} \\ 0 \end{pmatrix}$$

$$\begin{cases} \ddot{x} = \frac{qB_z}{m}\dot{y} \\ \ddot{y} = -\frac{qB_z}{m}\dot{x} \\ \ddot{z} = 0 \end{cases} \xrightarrow{a = \frac{qB_z}{m}} \begin{cases} \ddot{x} = a\dot{y} \\ \ddot{y} = -a\dot{x} \\ z = v_{z0}t + z_0 \end{cases}$$

$\ddot{z}$ can be integrated directly $\rightarrow \dot{z} = C = v_{z0}$

$$\begin{cases} \ddot{x} = a\dot{y} \\ \ddot{y} = -a\dot{x} \end{cases} \xrightarrow{d/dt} \begin{cases} \dddot{x} = a\ddot{y} \\ \dddot{y} = -a\ddot{x} \end{cases}$$

Let's write those as functions of v_x and v_y

$$\begin{cases} \dot{v_x} = av_y & (1) \\ \dot{v_y} = -av_x & (3) \end{cases} \rightarrow \begin{cases} \ddot{v}_x = a\dot{v}_y & (2) \\ \ddot{v}_y = -a\dot{v}_x & (4) \end{cases}$$

$$(2) + (3) \rightarrow -av_x = \frac{1}{a}\ddot{v}_x$$
$$(1) + (4) \rightarrow av_y = -\frac{1}{a}\ddot{v}_y$$
$$\ddot{v}_x = -a^2 v_x$$
$$\ddot{v}_y = -a^2 v_y$$

Harmonic oscillators!

Superposition of sin and cos:

$$v_x = A\cos(at) + B\sin(at)$$
$$v_y = A'\cos(at) + B'\sin(at)$$
$$x = \int v_x dt = \frac{A}{a}\sin at - \frac{B}{a}\cos(at) + c$$
$$y = \int v_y dt = \frac{A'}{a}\sin at - \frac{B'}{a}\cos(at) + c'$$

We know that $\ddot{x} = a\dot{y}$ so:

$$-Aa\sin(at) + Ba\cos(at) = A'a\cos(at) + B'a\sin(at)$$
$$-A = B' = -v_{x0}$$

$$B = A' = v_{y0}$$
$$x_0 = x(t_0) = \frac{-v_{y0}}{a} + C$$
$$C = x_0 + \frac{v_{y0}}{a}$$
$$y_0 = \frac{v_{x0}}{a} + C'$$
$$C' = y_0 - \frac{v_{x0}}{a}$$

$$\Rightarrow \quad \begin{aligned} x &= \frac{v_{x0}}{a}\sin(at) - \frac{v_{y0}}{a}(\cos(at) - 1) + x_0 \\ y &= \frac{v_{y0}}{a}\sin(at) - \frac{v_{x0}}{a}(1 - \cos(at)) + y_0 \end{aligned}$$

Exercise 2 [Systems of Differential Equations (6 points)]

Homogeneous linear systems of differential equations with constant coefficients.

a) $\begin{cases} x' = 3x + 6y \\ y' = -2x - 3y \end{cases} \rightarrow \text{Matrix } \underbrace{\begin{pmatrix} x' \\ y' \end{pmatrix}}_{z'} = \underbrace{\begin{pmatrix} 3 & 6 \\ -2 & -3 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{z}$

Solve the eigenvalues-eigenvectors problem

- eigenvalues $\det(A - \lambda E) = 0$

  $\det(A - \lambda E) = \det \begin{pmatrix} 3-\lambda & 6 \\ -2 & -3-\lambda \end{pmatrix} = -(3+\lambda)(3-\lambda) + 12 = -9 + \lambda^2 + 12 = \lambda^2 + 3 = 0$

  $\lambda_{1,2} = \pm i\sqrt{3}$

- eigenvectors $(A - \lambda E)\vec{c}_1 = 0$

  $\lambda_1 = i\sqrt{3}$:

$$\begin{pmatrix} 3 - i\sqrt{3} & 6 \\ -2 & -3 - i\sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{cases} 3x_1 - i\sqrt{3}x_1 + 6y_1 = 0 \\ -2x_1 - 3y_1 - i\sqrt{3}y_1 = 0 \end{cases}$$
$$y_1 = -\frac{3 - i\sqrt{3}}{6}x_1$$
$$-2x_1 + 3\frac{3}{6}x_1 - \frac{3i\sqrt{3}}{6}x_1 + \frac{3i\sqrt{3}}{6}x_1 + \frac{3}{6}x_1 = -2x_1 + 2x_1 = 0$$

$$x_1 = 1$$
$$y_1 = -\frac{3 - i\sqrt{3}}{6}$$

$\lambda_2 = -i\sqrt{3}$:

$$\begin{pmatrix} 3 + i\sqrt{3} & 6 \\ -2 & -3 + i\sqrt{3} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\begin{cases} 3x_2 + i\sqrt{3}x_2 + 6y_2 = 0 \\ -2x_2 - 3y_2 + i\sqrt{3}y_2 = 0 \end{cases}$$
$$y_2 = -\frac{3 + i\sqrt{3}}{6}x_2$$
$$-2x_2 + \frac{3}{2}x_2 + \frac{i\sqrt{3}}{2}x_2 - \frac{i\sqrt{3}}{2} + \frac{3}{6}x_2 = 0$$
$$-2x_2 + 2x_2 = 0$$
$$x_2 = 1$$
$$y_2 = -\frac{3 + i\sqrt{3}}{6}$$

$$\lambda_1 = i\sqrt{3}:$$
$$\vec{c}_1 = \begin{pmatrix} 1 \\ -\frac{3-i\sqrt{3}}{6} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + i \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{6} \end{pmatrix}$$
$$\lambda_2 = -i\sqrt{3}:$$
$$\vec{c}_2 = \begin{pmatrix} 1 \\ -\frac{3+i\sqrt{3}}{6} \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + i \begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{6} \end{pmatrix}$$

- Real solutions

$$\vec{z}_1 = \begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} = \vec{c}_1 e^{\lambda_1 t} = \begin{pmatrix} 1 \\ -\frac{3-i\sqrt{3}}{6} \end{pmatrix} e^{i\sqrt{3}t}$$
$$= \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \cos(\sqrt{3}t) - \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{6} \end{pmatrix} \sin(\sqrt{3}t) + i \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{6} \end{pmatrix} \cos(\sqrt{3}t) + i \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \sin(\sqrt{3}t)$$
$$\vec{z}_2 = \begin{pmatrix} x_2(t) \\ y21(t) \end{pmatrix} = \vec{c}_2 e^{\lambda_2 t} = \begin{pmatrix} 1 \\ -\frac{3+i\sqrt{3}}{6} \end{pmatrix} e^{-i\sqrt{3}t}$$
$$= \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \cos(\sqrt{3}t) + \begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{6} \end{pmatrix} \sin(\sqrt{3}t) + i \begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{6} \end{pmatrix} \cos(\sqrt{3}t) - i \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \sin(\sqrt{3}t)$$
$$z_{Re} = Re[a\vec{z}_1 + b\vec{z}_2] = \left[\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} \cos(\sqrt{3}t) - \begin{pmatrix} 0 \\ \frac{\sqrt{3}}{6} \end{pmatrix} \sin(\sqrt{3}t) \right] (a + b)$$

$$z_{Im} = Im[a\vec{z}_1 + b\vec{z}_2] = (a-b)\left[\begin{pmatrix} 0 \\ -\frac{\sqrt{3}}{6} \end{pmatrix}\cos(\sqrt{3}t) + \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}\sin(\sqrt{3}t)\right]$$

b) $\begin{cases} x' = 8x + y \\ y' = -4x + 4y \end{cases}$ $\quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 8 & 1 \\ -4 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \end{pmatrix}$

- Eigenvalues: $\det(A - \lambda E) = 0$

  $\det(A-\lambda E) = \det\begin{pmatrix} 8-\lambda & 1 \\ -4 & 4-\lambda \end{pmatrix} = (8-\lambda)(4-\lambda) + 4 = 32 - 8\lambda - 4\lambda + \lambda^2 + 4 =$

  $\lambda^2 - 12\lambda + 36 = (\lambda - 6)^2 = 0$

  $\rightarrow \quad \lambda_{1,2} = 6$

- Eigenvectors: $(A - \lambda E)\vec{c}_1 = 0 \rightarrow \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

  $\begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

  $2x_1 + y_1 = 0$

  $\vec{c} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

  first solution: $\vec{z}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{6t}$

  let's look for another solution of this kind $\vec{z}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} a + bt \\ c + dt \end{pmatrix} e^{6t}$

$$\vec{z}_2' = \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} = \begin{pmatrix} 6a + 6bt + b \\ 6c + 6dt + d \end{pmatrix} e^{6t} = A\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 8 & 1 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} a + bt \\ c + dt \end{pmatrix} e^{6t}$$

$$\begin{pmatrix} 6a + 6bt + b \\ 6c + 6dt + d \end{pmatrix} = \begin{pmatrix} 8a + 8bt + c + dt \\ -4a - 4bt + 4c + 4dt \end{pmatrix}$$

$$\begin{cases} 6a + b = 8a + c \\ 6bt = (8b + d)t \\ 6c + d = -4a + 4c \\ 6dt = (-4b + 4d)t \end{cases}$$

$$\begin{cases} -2a = c - b \\ -2b - d = 0 \qquad \rightarrow b = -\frac{d}{2} \\ 2c + d + 4a = 0 \\ 2d + 4b = 0 \end{cases}$$

$$
\begin{aligned}
c &= b - 2a \\
b &= 1 \\
d &= -2 \\
a &= 1 \\
c &= -1 \\
\vec{z}_2 &= \begin{pmatrix} 1+t \\ -1-2t \end{pmatrix} e^{6t}
\end{aligned}
$$

There is another way of retrieving this solution, which gives you an automatic choice of one of the vectors in the generalized eigenspace, which does not lie in the eigenspace itself. Such a vector is called generalized eigenvector and has to satisfy:

$$(A - \lambda\mathbb{I})^2 v_2 = 0 \quad \text{and} \quad (A - \lambda\mathbb{I}) v_2 \neq 0$$

The equality assures, that the vector v_2 lies in the generalized eigenspace ("Hauptraum" in german) and the inequality makes certain, that it is not an eigenvector.

If one solves $(A - \lambda\mathbb{I})v_2 = v_E$, with $v_E \in Eig(A; \lambda)$, this is automatically satisfied.

$$\textit{proof:} \quad (A - \lambda\mathbb{I})^2 v_2 = (A - \lambda\mathbb{I}) v_E = 0 \quad \text{and} \quad (A - \lambda\mathbb{I}) v_2 = v_E \neq 0 \quad \square$$

$$\begin{pmatrix} 2 & 1 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \Rightarrow \quad y_2 = 1 - 2x_2 \quad \Rightarrow \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ is a solution.}$$

A second solution to the differential equation is now given by: $z_2(t) = (v_2 + v_E t)e^{\lambda t}$

proof: $\quad z_2'(t) = [v_E + \lambda(v_2 + v_E t)]\, e^{\lambda t}$

$$Az_2(t) = [(v_E + \lambda v_2) + \lambda v_E t]\, e^{\lambda t}, \text{ using } (A - \lambda\mathbb{I})v_2 = v_E \Rightarrow Av_2 = v_E + \lambda v_2 \quad \square$$

Thus a general solution is given by:

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = a \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{6t} + b \begin{pmatrix} 1+t \\ -1-2t \end{pmatrix} e^{6t}$$

$$a, b \in \mathbb{R}$$

c) $\begin{cases} x' = x - y + 2z \\ y' = -x + y + 2z \\ z' = x + y \end{cases} \qquad \vec{z}' = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix} \vec{z}$

- Eigenvalues:

$$\det(A-\lambda E) = \det\begin{pmatrix}1-\lambda & -1 & 2\\ -1 & 1-\lambda & 2\\ 1 & 1 & 0-\lambda\end{pmatrix} = -2-2(1-\lambda)-2(1-\lambda)-2-\lambda(1-\lambda)^2$$
$$= -4-2+2\lambda-2+2\lambda-\lambda-\lambda^3+2\lambda^2+\lambda$$
$$= -\lambda^3+2\lambda^2+4\lambda-8 = 4(\lambda-2)+(-\lambda^2)(\lambda-2)$$
$$= (4-\lambda^2)(\lambda-2) = -(\lambda-2)^2(\lambda+2)$$
$$\lambda_{1,2} = 2$$
$$\lambda_3 = -2$$

- Eigenvector for $\lambda = -2$:

$$\begin{pmatrix}3 & -1 & 2\\ -1 & 3 & 2\\ 1 & 1 & 2\end{pmatrix}\begin{pmatrix}x_1\\ y_1\\ z_1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$
$$3x_1 - y_1 + 2z_1 = 0$$
$$-x_1 + 3y_1 + 2z_1 = 0$$
$$x_1 + y_1 + 2z_1 = 0$$
$$3x_1 - y_1 = -x_1 + 3y_1 \to x_1 = y_1$$
$$x_1 = -z_1$$
$$\vec{c}_1 = \begin{pmatrix}1\\1\\-1\end{pmatrix}$$

first solution: $\vec{Z}_1 = \begin{pmatrix}x_1(t)\\ y_1(t)\\ z_1(t)\end{pmatrix} = \begin{pmatrix}1\\1\\-1\end{pmatrix} e^{-2t}$

- Eigenvectors for $\lambda = 2$:

$$\begin{pmatrix}-1 & -1 & 2\\ -1 & -1 & 2\\ 1 & 1 & -2\end{pmatrix}\begin{pmatrix}x_2\\ y_2\\ z_2\end{pmatrix} \to \begin{pmatrix}1 & 1 & -2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$
$$x_2 + y_2 - 2z_2 = 0$$
$$\vec{c}_2 = \begin{pmatrix}1\\-1\\0\end{pmatrix}, \qquad \vec{c}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$$
$$\vec{Z}_2 = \begin{pmatrix}1\\-1\\0\end{pmatrix} e^{2t}, \qquad \vec{Z}_3 = \begin{pmatrix}1\\1\\1\end{pmatrix} e^{2t}$$

$$\vec{Z}(t) = a \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} e^{-2t} + b \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{2t} + c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}$$

d) $\begin{cases} x' = -x + y - z \\ y' = 2x - y + 2z \\ z' = 2x + 2y - z \end{cases} \qquad \Rightarrow \qquad \vec{Z}' = A\vec{Z},\ A = \begin{pmatrix} -1 & 1 & -1 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}$

- Eigenvalues

$$\det(A - \lambda E) = 0$$

$$\det \begin{pmatrix} -1-\lambda & 1 & -1 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{pmatrix} = -(1+\lambda)(\lambda-1)(\lambda+3) = 0$$

$$\lambda_1 = -1$$
$$\lambda_2 = -3$$
$$\lambda_3 = 1$$

- $\lambda_1 = -1$:

$$\begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$y_1 - z_1 = 0$$
$$2x_1 + 2z_1 = 0$$
$$2x_1 + 2y_1 = 0$$
$$y_1 = z_1$$
$$x_1 = -y_1$$

$$\vec{c}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \qquad \Rightarrow \qquad \vec{Z}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} e^{-t}$$

- $\lambda_2 = -3$:

$$\begin{pmatrix} 2 & 1 & -1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$
\begin{aligned}
2x_2 + y_2 - z_2 &= 0 \\
2x_2 + 2y_2 + 2z_2 &= 0 \\
y_2 - z_2 &= 2y_2 + 2z_2 \\
y_2 &= -3z_2 \\
2x_2 - 3z_2 - z_2 &= 0 \\
x_2 &= 2z_2
\end{aligned}
$$

$$
\vec{c}_2 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \qquad \Rightarrow \quad \vec{Z}_2 = \begin{pmatrix} x_2(t) \\ y_2(t) \\ z_2(t) \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} e^{-3t}
$$

- $\lambda_3 = 1$:

$$
\begin{pmatrix} -2 & 1 & -1 \\ 2 & -2 & 2 \\ 2 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
-2x_3 + y_3 - z_3 &= 0 \\
2x_3 - 2y_3 + 2z_3 &= 0 \\
2x_3 + 2y_3 - 2z_3 &= 0 \\
2y_3 - 2z_3 &= -2y_3 + 2z_3 \\
y_3 &= z_3 \\
x_3 - y_3 + y_3 &= 0
\end{aligned}
$$

$$
\vec{c}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \qquad \Rightarrow \quad \vec{Z}_3 = \begin{pmatrix} x_3(t) \\ y_3(t) \\ z_3(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{t}
$$

$$
\vec{Z}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = a \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} e^{-t} + b \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} e^{-3t} + c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} e^{t}
$$